Some fractional differential equations on mathematical biology

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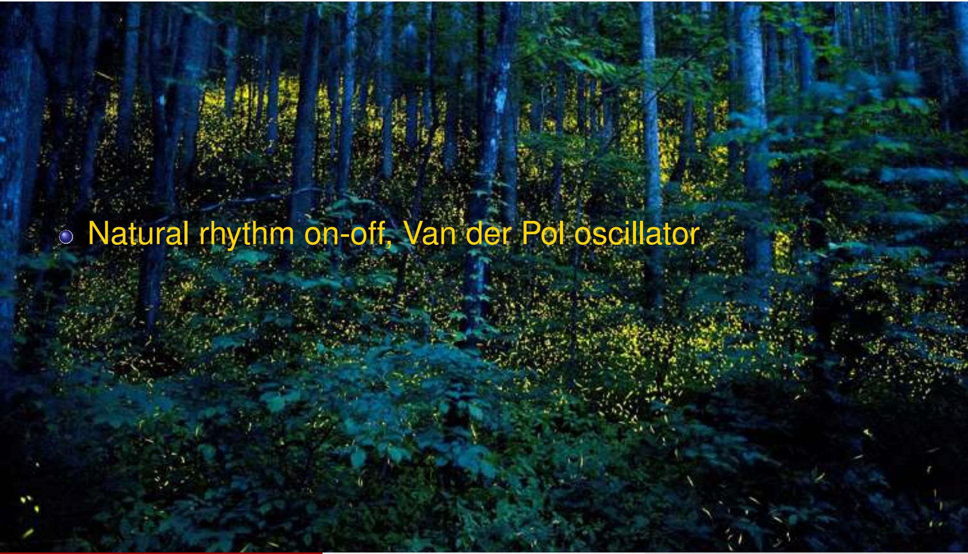
4th Mexican Workshop in Fractional Calculus

BIOLOGICAL ASSUMPTIONS AND CONSIDERATIONS OF THE MATHEMATICAL MODEL

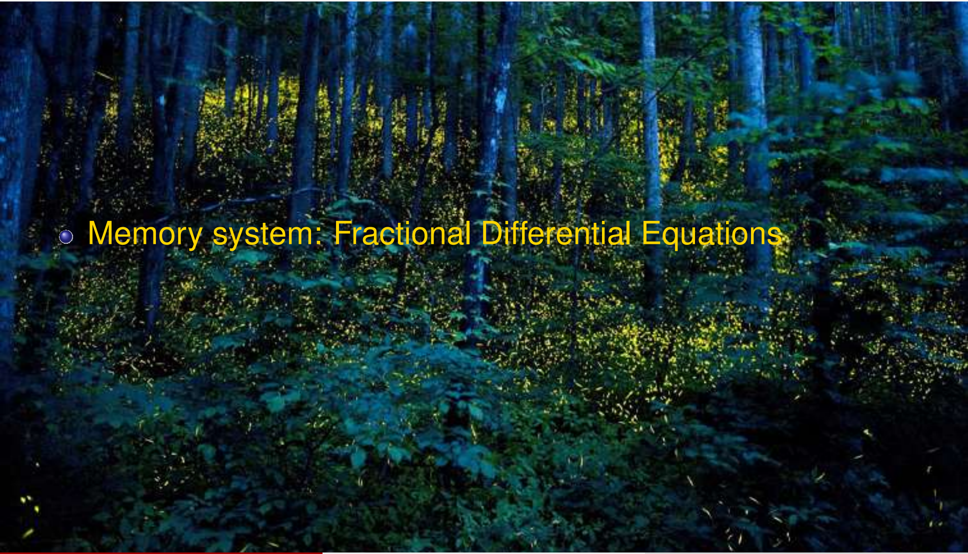


- Natural rhythm on-off, Van der Pol oscillator

BIOLOGICAL ASSUMPTIONS AND CONSIDERATIONS OF THE MATHEMATICAL MODEL

- Number of competitors and distance to them

BIOLOGICAL ASSUMPTIONS AND CONSIDERATIONS OF THE MATHEMATICAL MODEL



- Memory system: Fractional Differential Equations

$$\begin{aligned} \ddot{x}_i &= f_i(x_i, \dot{x}_i) + \sum_{j \neq i} a_{ij} x_j, \quad (1 \leq i, j \leq 4), \\ f_i(x_i, \dot{x}_i) &= \mu(1 - x_i^2)\dot{x}_i - a_{ii} x_i. \end{aligned} \tag{1}$$

The parameter a_{ij}, is the influence firefly i experiencing due to the brightness of the firefly j. Let be $a_{ii} = k$.

$$\begin{aligned}\ddot{x}_i &= f_i(x_i, \dot{x}_i) + \sum_{j \neq i} a_{ij} x_j, \quad (1 \leq i, j \leq 4), \\ f_i(x_i, \dot{x}_i) &= \mu(1 - x_i^2)\dot{x}_i - a_{ii} x_i.\end{aligned} \tag{1}$$

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The system (1) is expressed in a canonical form using the change of variables as follow: $z_1 = x_1$, $z_2 = \dot{x}_1$, $z_3 = x_2$, $z_4 = \dot{x}_2$, $z_5 = x_3$, $z_6 = \dot{x}_3$, $z_7 = x_4$ and $z_8 = \dot{x}_4$, now the system is given by:

$$\begin{aligned}&\dot{z}_{2i-1} = z_{2i}, \quad \dot{z}_{2i} = f_{2i-1}(z_{2i-1}, z_{2i}) + \sum_{\substack{s \neq i \\ 2j-1 \neq 2i-1}} a_{is} z_{2j-1}, \\ &1 \leq s, i, j \leq 4.\end{aligned} \tag{2}$$

SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

$$ {}_0^C D_t^\gamma z_{2i-1} = z_{2i}, \quad {}_0^C D_t^\gamma z_{2i} = f_{2i-1}(z_{2i-1}, z_{2i}) + \sum_{\substack{s \neq i \\ 2j-1 \neq 2i-1}} a_{is} z_{2j-1}, \tag{3} $$

$$ 1 \leq s, i, j \leq 4. $$

The Liouville-Caputo operator defining the fractional derivative for $(\gamma > 0)$ as

$$ {}_0^C \mathcal{D}_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(\eta)}{(t-\eta)^{\gamma-n+1}} d\eta, \tag{4} $$

where ${}_0^C \mathcal{D}_t^\gamma$ is a Liouville-Caputo fractional derivative with respect to t.

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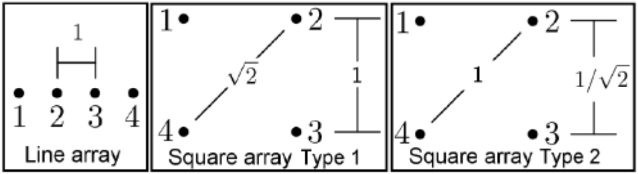

LINEAR ARRAY. ONE OR TWO COMPETITORS

System linearization (3) around of the equilibrium point $0 \in \mathbb{R}^8$ in terms of the parameters k, μ is defined by M_1

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -k & \mu & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -k & \mu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -k & \mu & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -k & \mu \end{bmatrix}$$

The eigenvalues of M_1 are $1/2(\mu \pm \sqrt{\pm 2 \pm \sqrt{5} - 4k + \mu^2})$. Since a periodic phenomenon modeled complex eigenvalues are needed and therefore

$$k > max\left\{\frac{1}{4}(\mu^2 \pm 2 \pm 2\sqrt{5})\right\}$$
$$= \frac{1}{4}(\mu^2 + 2 + 2\sqrt{5})$$
$$\approx 1.868033989.$$

SQUARE ARRAY. TYPE 1

System linearization (3) around of the equilibrium point $0 \in \mathbb{R}^8$ in terms of the parameters k, μ is defined by M_2

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -k & \mu & 1 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -k & \mu & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 & -k & \mu & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1/2 & 0 & 1 & 0 & -k & \mu \end{bmatrix}$$

The eigenvalues of M_2 are $1/2(\mu \pm \sqrt{-6 - 4k + \mu^2})$, $1/2(\mu \pm \sqrt{10 - 4k + \mu^2})$ and $1/2(\mu \pm \sqrt{-2 - 4k + \mu^2})$ of algebraic multiplicity 2. Thus

$$k > max\left\{\frac{1}{4}(\mu^2 - 6), \frac{1}{4}(\mu^2 + 10), \frac{1}{4}(\mu^2 - 2)\right\} = \frac{1}{4}(\mu^2 + 10) = 2.75$$

SQUARE ARRAY. TYPE 2

System linearization (3) around of the equilibrium point $0 \in \mathbb{R}^8$ in terms of the parameters k, μ is defined by M_3

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -k & \mu & 1/2 & 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & -k & \mu & 1/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1/2 & 0 & -k & \mu & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1/2 & 0 & 1 & 0 & 1/2 & 0 & -k & \mu \end{bmatrix}$$

The eigenvalues of M_3 are $1/2(\mu \pm \sqrt{-4 - 4k + \mu^2})$ of multiplicity 2, $1/2(\mu \pm \sqrt{-4k + \mu^2})$ and $1/2(\mu \pm \sqrt{8 - 4k + \mu^2})$. Thus

$$k > max\left\{\frac{1}{4}(\mu^2 - 4), \frac{1}{4}(\mu^2), \frac{1}{4}(\mu^2 + 8)\right\} = \frac{1}{4}(\mu^2 + 8) = 2.25$$

RESULTS AND CONCLUSIONS

RESULTS

- For the linear arrangement, where the fireflies only see one or two fireflies, the value of $k = 1.868033989$.
- For the square array type 1, $k = 2.75$.
- For the square array type 2, $k = 2.25$.
- For the linear array, where each receives three, $k = 2.1322$.

CONCLUSIONS

The value of k means the effort that each firefly makes to shine, this value increases when the number of competitors increases (fixed distance). In addition, it decreases when the distance between them is smaller (number of fixed competitors)

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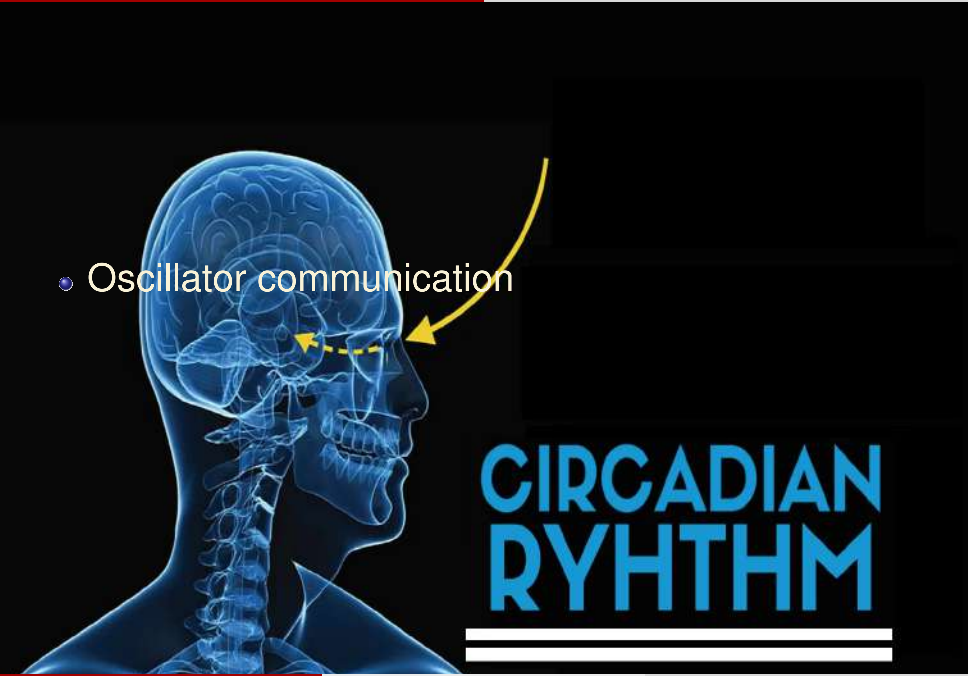
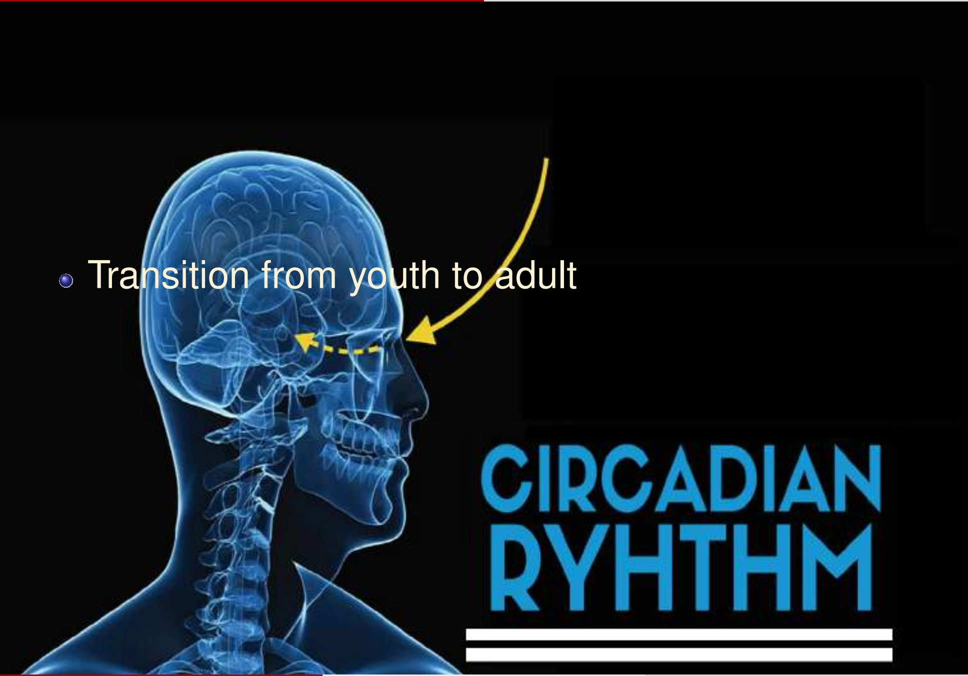
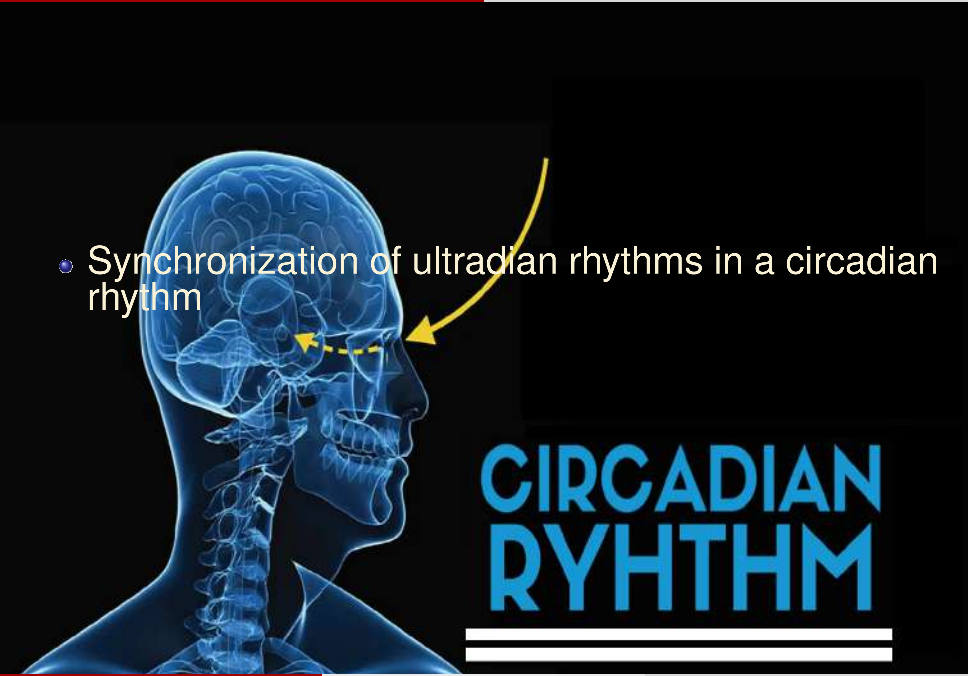
- Oscillator communication

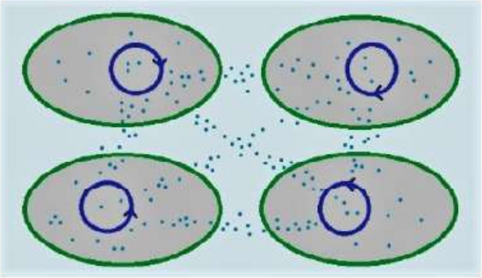


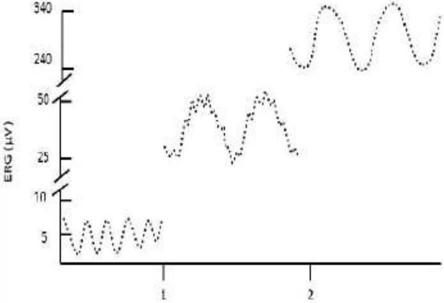


- Transition from youth to adult

- Synchronization of ultradian rhythms in a circadian rhythm






340
240
50
25
10
5
ERG (μV)
1
2

μ, IT IS A PARAMETER OF COMMUNICATION EFFICIENCY BETWEEN RHYTHMS

$$
\begin{aligned}
\ddot{x}_1 &= \nu(1-x_1^2)\dot{x}_1 - kx_1 + \mu(a_{12}x_2 + a_{13}x_3 + a_{14}x_4),\\
\ddot{x}_2 &= \nu(1-x_2^2)\dot{x}_2 - kx_2 + \mu(a_{21}x_1 + a_{23}x_3 + a_{24}x_4),\\
\ddot{x}_3 &= \nu(1-x_3^2)\dot{x}_3 - kx_3 + \mu(a_{31}x_1 + a_{32}x_2 + a_{34}x_4),\\
\ddot{x}_4 &= \nu(1-x_4^2)\dot{x}_4 - kx_4 + \mu(a_{41}x_1 + a_{42}x_2 + a_{43}x_3).
\end{aligned} \tag{5}
$$

The coefficients a_{ij} in (5) they are given by the following matrix

$$
\begin{bmatrix}
-k & 1 & 1/\sqrt[4]{2} & 1\\
1 & -k & 1 & 1/\sqrt[4]{2}\\
1/\sqrt[4]{2} & 1 & -k & 1\\
1 & 1/\sqrt[4]{2} & 1 & -k
\end{bmatrix}. \tag{6}
$$

The **Liouville-Caputo** fractional derivative of order γ is defined by

$$ {}_{a}^{LC}\mathcal{D}_{t}^{\gamma}\{f(t)\} = \frac{1}{\Gamma(n-\gamma)} \int_{a}^{t} f^{(n)}(\theta)(t-\theta)^{n-\gamma-1} d\theta, \quad n-1 < \gamma \leq n. \tag{7} $$

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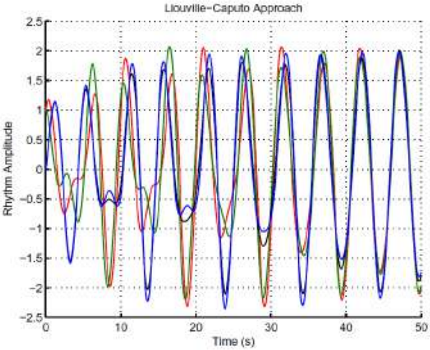
By changing the kernel $(t-\theta)^{-\gamma}$ in (7) with the function $\exp(-\frac{\gamma}{n-\gamma}t)$ and $\frac{1}{\Gamma(n-\gamma)}$ with $\frac{M(\gamma)}{n-\gamma}$, it is obtained the **Caputo-Fabrizio** fractional derivative:

$$ {}_{a}^{CF}\mathcal{D}_t^{\gamma}\{f(t)\} = \frac{M(\gamma)}{n-\gamma} \int_a^t f^{(n)}(\theta) \exp\left[-\frac{\gamma}{n-\gamma}(t-\theta)\right] d\theta, \quad n-1 < \gamma \leq n. \qquad (8) $$

Where $M(\gamma)$ is a normalization function such that $M(0) = M(1) = 1$.

RHYTHMS: ELECTRORETINOGRAPHY (ERG) VS STAGES OF CRAYFISH



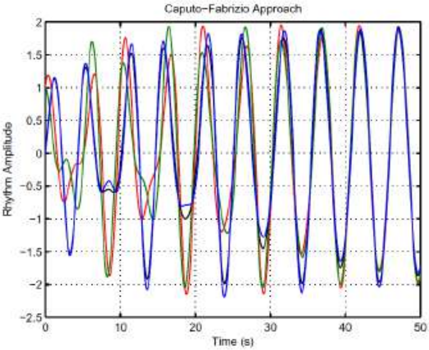


FIGURE: $\mu = 0.5$, $\gamma = 0.9, 0.85, 0.8, 0.75$.

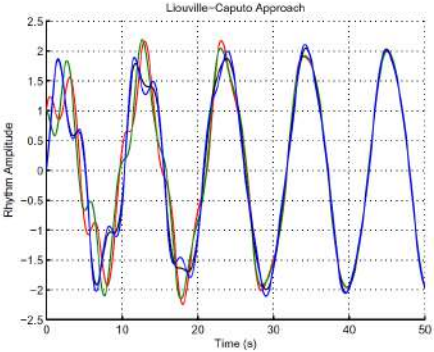


FIGURE: $\mu = 0.8$, $\gamma = 0.9, 0.85, 0.8, 0.75$.

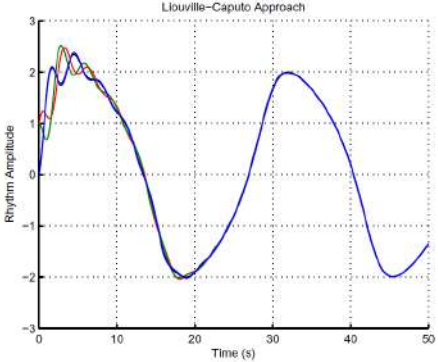


FIGURE: $\mu = 1$, $\gamma = 0.9, 0.85, 0.8, 0.75$.

CONCLUSIONS

- $0 < \mu < 0.7$. Ultradian rhythms shown at this stage are clearly out of synch.
- $0.7 < \mu < 1$. A low frequency rythm appears on which ultradian rhythms can be observed that correspond to stage 1-2.
- $\mu = 1$. The crayfish is mature, the circadian rhythm is more apparent. In fact, ultradian rhythms are synchronized in a single circadian rhythm responsible for governing the metabolism of the adult crayfish.

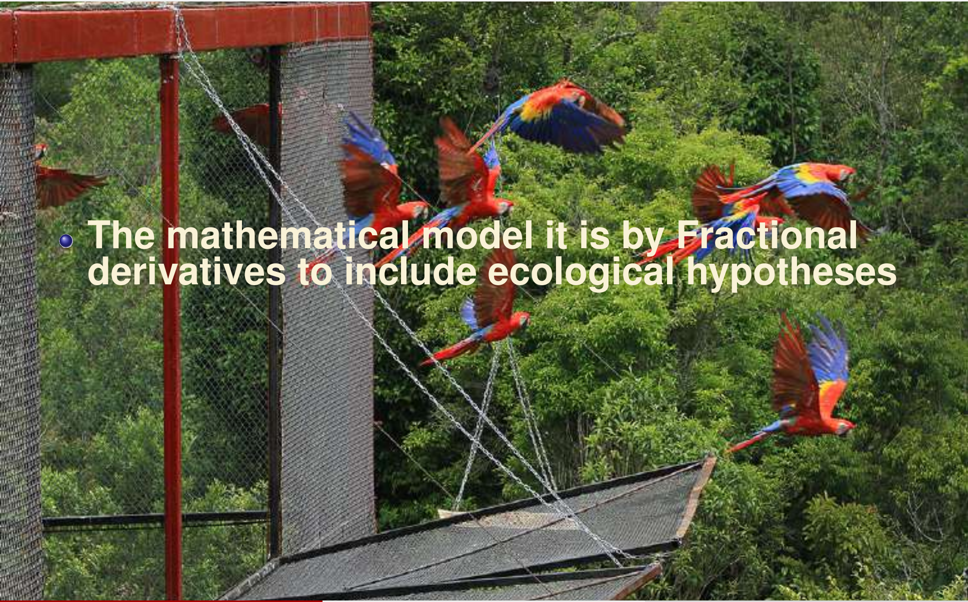
Dynamics in Natural time

- **Interaction of three species: Pollinators, plants, herbivores**

Dynamics in Natural time

- **Persistence of a Hopf bifurcation**

DYNAMICS IN NATURAL TIME



- The mathematical model it is by Fractional derivatives to include ecological hypotheses

DEFINITION

Caputo fractional derivative with order α for a function $f \in \mathcal{AC}^m[0,T],\ T > 0$ with absolutely continuos derivative up to order $m-1$ and with absolutely continuos m-derivative is defined as

$$ {}^{C}D^{\alpha}_{t_0} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau. \tag{9} $$

where $0 \leq m-1 < \alpha \leq m,\ m \in \mathbb{Z}^+$, and $t = t_0$ is the initial time and $\Gamma(\cdot)$ is the Gamma function. $f^{(m)}$ is the derivative of f of order m.

To keep the dimensionality of the system of fractional differential equations, we introduce a parameter ν, named "natural time", in the following way. **The specific value that the natural time takes depends on each investigation, this should be a unit defined by the own biological rhythms of the studied system.**

$$\begin{aligned} \frac{df}{dt} &\simeq \frac{1}{\nu^{1-\alpha}} \frac{d^\alpha f}{dt^\alpha}, \\ &= \frac{1}{\nu^{1-\alpha}} \, {}^C D_{t_0}^\alpha f(t), \end{aligned} \qquad m-1 < \alpha \leq m, \quad m \in \mathbb{Z}^+. \tag{10}$$

SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

The system is derived on some plausible and widely supported ecological hypothesis. Respectively: x-pollinator, y-plants and z-herbivores.

$$\begin{aligned}
\frac{1}{\nu^{1-\alpha}}\frac{d^\alpha x}{dt^\alpha} &= bx(k-x) + \frac{g(z)k_2\sigma\mu^2 xy}{1+\phi\sigma\mu^2 y}, \\
\frac{1}{\nu^{1-\alpha}}\frac{d^\alpha y}{dt^\alpha} &= \frac{g(z)k_1\sigma\mu xy}{1+\phi\sigma\mu^2 y} - \gamma y - \frac{m_1 yz}{\frac{y^2}{c}+y+a}, \\
\frac{1}{\nu^{1-\alpha}}\frac{d^\alpha z}{dt^\alpha} &= \frac{m_2 yz}{\frac{y^2}{c}+y+a} - \delta z.
\end{aligned} \tag{11}$$

where $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$, and $g(z)$ measures the reduction rate of pollinator visits, which depends on the herbivore population density

STABILITY ANALYSIS

$M(x, yz) =$

$$\begin{bmatrix} b(k-2x) + \mu k_2 y h_2(y) & \mu k_2 x h_3(y) & 0 \\ k_1 y h_2(y) & k_1 x h_3(y) + m_1 y(2y+c) h_6(y) - m_1 z h_5(y) - \gamma & -m_1 y h_5(y) \\ 0 & m_2 z h_5(y) - m_2 y z (2y+c) h_6(y) & -m_2 y h_5(y) - \delta \end{bmatrix}$$

with $h_1(y) = y\sigma\phi\mu^2 + 1$, $h_2(y) = \dfrac{g(z)\sigma\mu}{h_1(y)}$, $h_3(y) = \dfrac{h_2(y)}{h_1(y)}$,
$h_4(y) = y^2 + cy + ac$, $h_5(y) = \dfrac{c}{h_4(y)}$, $h_6(y) = \dfrac{h_5(y)}{h_4(y)}$.

TABLE: Equilibrium points of the system (11)

x	y	z	Plausible
0.6666	-2.0	0	$\times$
0	22.4888	-187.4066	$\times$
0	0.1778	-1.4822	$\times$
0.9664	-0.1141	0	$\times$
2.9183	0.1778	1.0702	$\checkmark$
0	0	0	$\times$
5.4238	22.4888	-169.7203	$\times$
0	-0.5	0	$\times$
2	0	0	$\times$
0	-2.0	0	$\times$
4.5353	1.3141	0	$\times$

$$N(p) = \begin{bmatrix} -3.7695 & 14.3579 & 0 \\ 0.3388 & -1.0776 & -0.3874 \\ 0 & 2.9263 & -0.0002 \end{bmatrix} \quad (12)$$

$$\begin{aligned} \lambda_1 &= -4.9544 \\ \lambda_2 &= 0.0536 + 0.9271\, i \\ \lambda_2 &= 0.0536 - 0.9271\, i \end{aligned} \quad (13)$$

When $\alpha \in (0, 0.9625],\ p$ is a spiral stable. For $\alpha \in (0.9625, 1]$ system has a limit cycle.

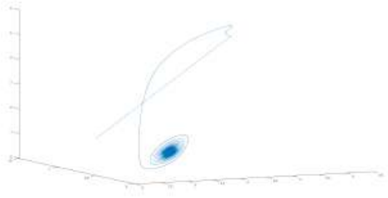


FIGURE: Spiral

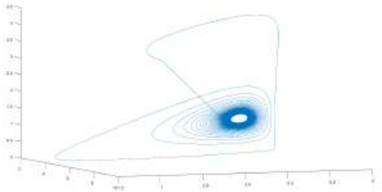


FIGURE: Limit Cycle

CONCLUSIONS

- Stability in both cases is plausible from the ecological perspective.
- Varying the order of derivation causes a Hopf fork to appear.
- The derivation orders can be different between equations in the numerical method used.

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Thank you
very much
for you attention