APPLICATION OF SERIES METHOD WITH PADÉ AND LAPLACE-PADÉ RESUMMATION METHODS TO SOLVE A MODEL FOR THE EVOLUTION OF SMOKING HABIT IN SPAIN

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Abstract. We obtain approximated analytical solutions of a mathematical model of the evolution of smoking habit in Spain [1] using the series method. To enlarge the domain of convergence, we apply the Padé and Laplace-Padé resummation methods to the series solution. We present a comparison of our results and a solution obtained using homotopy analysis method [2], resulting that the combination of series method with Laplace-Padé resummation method generates the best results for the complete domain.

Mathematical subject classification: 34L30

Smoking habit, Series method, Padé, Laplace-Padé.

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1. Introduction

The fact that epidemic models consist of a system of non-linear differential equations underlines the importance of having reliable methods for solving them. This type of models can be integrated using any standard numerical method. However, it is known that these algorithms can give some problems, such as numerical instabilities, oscillations or false equilibrium states, among others. This means that the numerical solution may not correspond to the real solution of the original system of differential equations [3]. This is the reason why we are interested in obtaining a continuous solution in the form of an analytical approximation to the real solution. Among the different methods that have been developed to obtain analytical approximations for the solution of a system of ODEs, we can use the series method (SM) [4, 5, 6], homotopy perturbation method [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20], homotopy analysis method [2, 21], among others.

The epidemic model for smoking habit in Spain is a system of non-linear differential equations without closed solution [1]. The interest of this model is that it has been able to describe correctly the real evolution of the spread of the smoking habit in Spain. It was constructed using real data for the initial values and for the parameters of the system. Constant population is assumed by taking birth and death rates equal and different from zero.

In this work, we propose the use of the series method to obtain an analytical approximation to the solution smoking habit model. Moreover, in order to enlarge the domain of the power series, we propose the use of Padé and Laplace-Padé resummation methods. The proposed solutions are compared with a HAM solution reported in [2], resulting that the solution obtained with the series method and treated with Laplace-Padé method generates the most accurate reported approximated solution valid for $t = [0, \infty]$ years.

This paper is organized as follows. In Section 2, we provide a brief review of the series method. Section 3 presents the basic concept of Padé and Laplace-Padé resummation methods. In Section 4, we introduce the mathematical model of the evolution of smoking habit in Spain. After that, we present the solution obtained with the series method for the smoking habit model in Section 5. In Section 6, we present the resulting power series solution. Next, Section 7 shows results and discuss our findings. Finally, a concluding remark is given in Section 8.

2. Basic concept of series method

It can be considered that a nonlinear differential equation can be expressed as

$$A(u) - f(t) = 0, \tag{1}$$

where A is a general operator and $f(t)$ is a known analytic function of independent variable t

The series method establishes that the solution of a differential equation can be written as

$$u = \sum_{i=0}^{\infty} u_i t^i, \tag{2}$$

where $u_0, u_1, \ldots$ are unknowns to be determined by series method.

The basic process of series method can be described as:

(1) Equation (2) is substituted into (1), then we regroup equation in terms of t-powers.
(2) We equate each coefficient of the resulting polynomial to zero.
(3) The boundary conditions of (1) are substituted into (2) to obtain an approximation for each initial condition.
(4) Aforementioned steps generates a nonlinear algebraic equation system (NAEs) in terms of the unknowns of (2).
(5) Finally, we solve the NAEs to obtain $u_0, u_1, \ldots$, coefficients.

3. Padé and Laplace-Padé resummation methods

Several approximated methods provide power series solutions (polynomial). Nevertheless, sometimes, this type of solutions lacks of large domains of convergence. Therefore, Padé [2, 22, 23, 24] and Laplace-Padé [25, 26, 27, 28, 29, 30, 31, 32, 33, 34] resummation methods are used in literature to enlarge the domain of convergence of solutions.

On one side, the Padé resummation method consists of applying the Padé approximant of order $[N/M]$ to the power series solution. N and M are arbitrarily chosen, but they should be of smaller value than the order of the power series. On the other side, the Laplace-Padé resummation method for power series solutions can be recast as follows:

(1) First, Laplace transformation is applied to power series (2).
(2) Next, s is substituted by $1/t$ in the resulting equation.
(3) After that, we convert the transformed series into a meromorphic function by forming its Padé approximant of order $[N/M]$. N and M are arbitrarily chosen, but they should be of smaller value than the order

of the power series. In this step, the Padé approximant extends the domain of the truncated series solution to obtain better accuracy and convergence.

(4) Then, t is substituted by $1/s$.
(5) Finally, by using the inverse Laplace s transformation, we obtain the modified approximated solution.

This process is known as the Laplace-Padé series method (LPSM).

4. The model of the evolution of smoking habit in Spain

This model was presented in [1, 2] to describe and predict the evolution of the smoking habit in Spain and to quantify the impact of the Spanish smoke-free law of 2006.

The following system of ordinary differential equations models the dynamics between the different subpopulations considered.

$$\begin{aligned}
&\dot{n} - \mu(1-n) + \beta n(s+c) = 0,\\
&\dot{s} - \beta n(s+c) - \rho e - \alpha c + (\gamma+\lambda+\mu)s = 0,\\
&\dot{c} - \gamma s + (\alpha+\delta+\mu)c = 0,\\
&\dot{e} - \lambda s - \delta c + (\rho+\mu)e = 0,
\end{aligned} \tag{3}$$

where the dots denote differentiation with respect to t.

The subpopulations included in the model are: n is the proportion of the total population who has never smoked, s is the proportion of people who smoke less than 20 cigarettes per day, c is the proportion of individuals who smoke more than 20 cigarettes per day and e is the proportion of ex-smokers.

The parameter μ denotes birth rate in Spain; β denotes the transmission rate due to the social pressure to adopt smoking habit; ρ expresses the rate at which ex-smokers return to smoking; α is the rate at which an excessive smoker becomes a normal smoker by decreasing the number of cigarettes per day; γ is the rate at which normal smokers become excessive smokers by increasing the number of cigarettes per day; λ denotes the rate at which normal smokers stop smoking and δ is the rate at which excessive smokers stop smoking. The population is constant and it has been normalized to unity, then:

$$n + s + c + e = 1, \tag{4}$$

for any instant of time.

The asymptotic behaviour of (3) is obtained considering $\dot{n}=0$, $\dot{s}=0$, $\dot{c}=0$, and $\dot{e}=0$, and solving the obtained system of equations, resulting two equilibrium points: SFE (smoking free equilibrium) and SEE (smoking endemic equilibrium), which are $SFE=(1,0,0,0)$ and $SEE=(n^*_{SEE}, s^*_{SEE}, c^*_{SEE}, e^*_{SEE})$ where

(5)
$$n^*_{SEE} = \frac{\left(\mu^2+(\lambda+\alpha+\delta+\rho+\gamma)\,\mu+(\alpha+\delta+\gamma)\,\rho+(\gamma+\lambda)\,\delta+\lambda\alpha\right)\mu}{\beta\,(\alpha+\delta+\mu+\gamma)\,(\rho+\mu)},$$
$$s^*_{SEE} = \frac{\eta_2(\alpha+\delta+\mu)}{\eta_1},$$
$$c^*_{SEE} = \frac{\eta_2\gamma}{\eta_1},$$
$$e^*_{SEE} = \frac{\eta_2(\lambda\mu+(\gamma+\lambda)\delta+\lambda\alpha)}{\eta_1(\rho+\mu)},$$

with

(6)
$$\eta_1 = \Big[\mu^2+(\lambda+\alpha+\delta+\rho+\gamma)\,\mu + (\lambda+\rho+\gamma)\,\delta+\rho\gamma+\alpha\,(\lambda+\rho)\Big]\,(\alpha+\delta+\mu+\gamma)\,\beta,$$
$$\eta_2 = -\Big[\mu^3+(\lambda+\rho-\beta+\alpha+\delta+\gamma)\mu^2+\Big((\lambda-\beta+\gamma+\rho)\delta+(\rho-\beta)\gamma +(\lambda-\beta+\rho)\alpha-\beta\rho\Big)\mu-\beta\rho(\alpha+\delta+\gamma)\Big].$$

Studying the eigenvalues of the Jacobian matrix associated to the system (3) we can analyze which one of the two equilibrium points is asymptotically stable and which one is unstable. Moreover, since we are going to use the values of the parameters given in [2], we can conclude that the point SEE is the stable one. Therefore, our solution has to tend asymptotically to the SEE point. For more details about this question, see [35].

5. Approximated series solution

According to the series method, we propose the following solution of order m

$$n(t) = \sum_{k=0}^{m} n_k t^i, \qquad s(t) = \sum_{i=0}^{m} s_i t^i, \tag{7}$$
$$c(t) = \sum_{k=0}^{m} c_i t^i, \qquad e(t) = \sum_{i=0}^{m} e_i t^i,$$

where the initial approximations are

$$\begin{aligned} n_0 &= n(0), \\ s_0 &= s(0), \\ c_0 &= c(0), \\ e_0 &= e(0). \end{aligned} \tag{8}$$

Substituting (7) into (3), we obtain

(9)

$$\sum_{k=1}^{m-1} k n_k t^{k-1} = \mu\left(1 - \sum_{k=0}^{m} n_k t^k\right) - \beta\left(\sum_{k=0}^{m} n_k t^k\right)\left(\sum_{k=0}^{m}(s_k + c_k) t^k\right),$$

$$\sum_{k=1}^{m-1} k s_k t^{k-1} = \beta\left(\sum_{k=0}^{m} n_k t^k\right)\left(\sum_{k=0}^{m}(s_k + c_k) t^k\right) + \rho\sum_{k=0}^{m} e_k t^k + \alpha\sum_{k=0}^{m} c_k t^k - (\gamma + \lambda + \mu)\sum_{k=0}^{m} s_k t^k,$$

$$\sum_{k=1}^{m-1} k c_k t^{k-1} = \gamma\sum_{k=0}^{m} s_k t^k - (\gamma + \delta + \mu)\sum_{k=0}^{m} c_k t^k,$$

$$\sum_{k=1}^{m-1} k e_k t^{k-1} = \lambda\sum_{k=0}^{m} s_k t^k + \delta\sum_{k=0}^{m} c_k t^k - (\rho + \mu)\sum_{k=0}^{m} e_k t^k.$$

Next, we equate the terms of the same order, resulting

$$\begin{aligned} n_1 &= \mu(1 - n_0) - \beta n_0(s_0 + c_0), \\ s_1 &= \beta n_0(s_0 + c_0) + \rho e_0 + \alpha c_0 - (\gamma + \lambda + \mu) s_0, \\ c_1 &= \gamma s_0 - (\gamma + \delta + \mu) c_0, \\ e_1 &= \lambda s_0 + \delta c_0 - (\rho + \mu) e_0, \end{aligned} \tag{10}$$

and

$$\begin{aligned} 2n_2 &= -\mu n_1 - \beta\left[n_0(s_0 + c_0) + n_1(s_0 + c_0)\right], \\ 2s_2 &= \beta\left[n_0(s_0 + c_0) + n_1(s_0 + c_0)\right] + \rho e_1 + \alpha c_1 - (\gamma + \lambda + \mu) s_1, \\ 2c_2 &= \gamma s_1 - (\gamma + \delta + \mu) c_1, \\ 2e_2 &= \lambda s_1 + \delta c_1 - (\rho + \mu) e_1, \end{aligned} \tag{11}$$

for powers t_0 and t_1, respectively.

For the rest of coefficients, we find the following recursive formula

$$
\begin{aligned}
n_{k+1} &= \frac{1}{1+k}\left[-\mu n_k - \beta \sum_{p=k}^{m} n_p (s_{k-p} + c_{k-p})\right], \\
s_{k+1} &= \frac{1}{1+k}\left[\beta \sum_{p=k}^{m} n_p (s_{k-p} + c_{k-p}) + \rho e_k + \alpha c_k - (\gamma + \lambda + \mu) s_k\right], \\
c_{k+1} &= \frac{1}{1+k}\left[\gamma s_k - (\gamma + \delta + \mu) c_k\right], \\
e_{k+1} &= \frac{1}{1+k}\left[\lambda s_k + \delta c_k - (\rho + \mu) e_k\right],
\end{aligned} \tag{12}
$$

for $k \geq 1$.

Therefore, (12) allows us to obtain the series solution in powers of t for any order m.

6. Padé and Laplace-Padé resummation methods

In order to perform the resummation methods, we set the values of the parameters as in [2]: $\mu = 0.01$ years^{-1}, $\rho = 0.0425$ years^{-1}, $\beta = 0.0381$ years^{-1}, $\beta = 0.0381$ years^{-1}, $\alpha = 0.1244$ years^{-1}, $\gamma = 0.1175$ years^{-1}, $\lambda = 0.0498$ years^{-1} and $\delta = 0.0498$ years^{-1}. Moreover, the initial conditions are chosen as: $n(0) = 0.5045$, $s(0) = 0.2059$, $c(0) = 0.1559$ and $e(0) = 0.1337$, as reported in

[2]. Then, from (12), we obtain the following solution 20th order approximation (13)

$$\begin{aligned}
n(t) &= 0.50450000 - 0.0019993207t + 0.00011026408t^2 - 0.0000036817027t^3\\
&+0.000000096547532t^4 - 0.0000000022242078t^5 + 4.8437208\times10^{-11}t^6\\
&-1.0350119\times10^{-12}t^7 + 2.1771775\times10^{-14}t^8 - 4.4505566\times10^{-16}t^9\\
&+8.7339077\times10^{-18}t^{10} - 1.6342918\times10^{-19}t^{11} + 2.9017298\times10^{-21}t^{12}\\
&-4.8437592\times10^{-23}t^{13} + 7.4350914\times10^{-25}t^{14} - 9.9258933\times10^{-27}t^{15}\\
&+9.5586875\times10^{-29}t^{16} + 1.2390588\times10^{-31}t^{17} - 4.0553722\times10^{-32}t^{18}\\
&+1.4784579\times10^{-33}t^{19} - 4.0334550\times10^{-35}t^{20},
\end{aligned}$$

$$\begin{aligned}
s(t) &= 0.20590000 - 0.0044755390t + 0.00024884128t^2 - 0.000012284248t^3\\
&+0.00000063314928t^4 - 0.000000032767620t^5 + 0.0000000015631567t^6\\
&-6.6227760\times10^{-11}t^7 + 2.4834689\times10^{-12}t^8 - 8.3079136\times10^{-14}t^9\\
&+2.5041966\times10^{-15}t^{10} - 6.8647266\times10^{-17}t^{11} + 1.7252816\times10^{-18}t^{12}\\
&-4.0029155\times10^{-20}t^{13} + 8.6247171\times10^{-22}t^{14} - 1.7345687\times10^{-23}t^{15}\\
&+3.2708638\times10^{-25}t^{16} - 5.8059941\times10^{-27}t^{17} + 9.7357472\times10^{-29}t^{18}\\
&-1.5471402\times10^{-30}t^{19} + 2.3368646\times10^{-32}t^{20},
\end{aligned}$$

$$\begin{aligned}
c(t) &= 0.15590000 - 0.0045235300t + 0.00015367920t^2 + 0.00000031038033t^3\\
&-0.00000037514280t^4 + 0.000000028699268t^5 - 0.0000000015227668t^6\\
&+6.6309221\times10^{-11}t^7 - 2.4994901\times10^{-12}t^8 + 8.3579298\times10^{-14}t^9\\
&-2.5157106\times10^{-15}t^{10} + 6.8876090\times10^{-17}t^{11} - 1.7294192\times10^{-18}t^{12}\\
&+4.0098432\times10^{-20}t^{13} - 8.6353979\times10^{-22}t^{14} + 1.7360297\times10^{-23}t^{15}\\
&-3.2724281\times10^{-25}t^{16} + 5.8065162\times10^{-27}t^{17} - 9.7320256\times10^{-29}t^{18}\\
&+1.5455734\times10^{-30}t^{19} - 2.3324180\times10^{-32}t^{20},
\end{aligned}$$

$$\begin{aligned}
e(t) &= 0.13370000 + 0.010998390t - 0.00051278455t^2 + 0.000015655570t^3\\
&-0.00000035455400t^4 + 0.0000000062925616t^5 - 8.8827250\times10^{-11}t^6\\
&+9.5354971\times10^{-13}t^7 - 5.7505700\times10^{-15}t^8 - 5.5106120\times10^{-17}t^9\\
&+2.7801100\times10^{-18}t^{10} - 6.5396364\times10^{-20}t^{11} + 1.2357333\times10^{-21}t^{12}\\
&-2.0840615\times10^{-23}t^{13} + 3.2457857\times10^{-25}t^{14} - 4.6820917\times10^{-27}t^{15}\\
&+6.0836875\times10^{-29}t^{16} - 6.4611765\times10^{-31}t^{17} + 3.3288889\times10^{-33}t^{18}\\
&+8.8347368\times10^{-35}t^{19} - 4.1332500\times10^{-36}t^{20}.
\end{aligned}$$

In order to enlarge the domain of convergence, we apply a Padé approximant [36] to (13) to obtain

(14)

$$n(t)_{[8/8]} = \Big(0.50449998 + 0.017274274t - 0.00030937773t^2 - 0.000033327548t^3 - 0.00000097656272t^4 - 0.000000016082178t^5 - 0.00000000016936989t^6 - 1.1596923\times 10^{-12}t^7 - 4.3920197\times 10^{-15}t^8\Big)/\Delta_1,$$

$$\Delta_1 = 0.99999997 + 0.038203359t - 0.00068039852t^2 - 0.000069808996t^3 - 0.0000019762218t^4 - 0.000000032319349t^5 - 0.00000000033869281t^6 - 2.2556172\times 10^{-12}t^7 - 8.5053029\times 10^{-15}t^8,$$

$$s(t)_{[10/10]} = \Big(0.20590000 - 0.12171188t - 0.0074901950t^2 - 0.000036696032t^3 + 0.000021046910t^4 + 0.0000011319850t^5 + 0.000000039940480t^6 + 0.00000000071857987t^7 + 9.8768910\times 10^{-12}t^8 + 4.0691945\times 10^{-14}t^9 + 1.7800146\times 10^{-16}t^{10}\Big)/\Delta_2,$$

$$\Delta_2 = 1.0 - 0.56938483t - 0.049962802t^2 - 0.00051644392t^3 + 0.00011433093t^4 + 0.0000075362210t^5 + 0.00000024423564t^6 + 0.0000000047931141t^7 + 5.6947603\times 10^{-11}t^8 + 3.5636890\times 10^{-13}t^9 + 6.8478700\times 10^{-16}t^{10},$$

$$c(t)_{[7/7]} = \Big(0.15590000 + 0.015745924t + 0.00076149012t^2 + 0.000027274768t^3 + 0.00000056591323t^4 + 0.0000000097728885t^5 + 5.8352741\times 10^{-11}t^6 + 3.5093886\times 10^{-13}t^7\Big)/\Delta_3,$$

$$\Delta_3 = 0.99999999 + 0.13001574t + 0.0076712060t^2 + 0.00026738041t^3 + 0.0000059737020t^4 + 0.000000085942960t^5 + 0.00000000073959329t^6 + 2.9314471\times 10^{-12}t^7,$$

$$e(t)_{[7/7]} = \Big(0.13370000 + 0.016494265t + 0.0000086701166t^2 - 0.00000021344924t^3 - 0.000000044755062t^4 - 0.00000000087169957t^5 - 4.1776825\times 10^{-12}t^6 - 2.4269335\times 10^{-14}t^7\Big)/\Delta_4,$$

$$\Delta_4 = 1.0 + 0.041106021t + 0.00051873320t^2 - 0.0000037076810t^3 - 0.00000020166319t^4 - 0.0000000029490808t^5 - 2.2606150\times 10^{-11}t^6 - 8.7346224\times 10^{-14}t^7,$$

where different orders of Padé approximants are required in order to obtain the best accuracy for each one of the variables.

Now, we apply the Laplace-Padé resummation method to $n(t)$ from (13). First, Laplace transformation is applied to (13) and then $1/t$ is written in place of s in the equation. Afterwards, Padé approximant ([6/5] for $n(t)$ and $s(t)$, [5/4] for $c(t)$ and $e(t)$) is applied and $1/s$ is written in place of t. Finally, by using the inverse Laplace s transformation, we obtain the modified approximated solution

$$
\begin{aligned}
n(t)_{[6/5]} = {} & 0.51086151 \\
& -0.59911800 \times 10^{-4} \exp(-0.21704764t) \cos(0.034646335t) \\
& +0.83968340 \times 10^{-4} \exp(-0.21704764t) \sin(0.034646335t) \\
& +0.47169326 \times 10^{-3} \exp(-0.20129121t) \\
& +0.024786891 \exp(-0.091934389t) \\
& -0.031560192 \exp(-0.011358812t),
\end{aligned} \tag{15}
$$

Next, we apply the same procedure for the rest of the variables of (13), resulting

$$
\begin{aligned}
s(t)_{[6/5]} = {} & 0.15307771 + 0.001406921 \exp(-0.30192977t) \\
& +0.57701934 \times 10^{-4} \exp(-0.28788644t) \\
& +0.037344784 \exp(-0.09336449t) \\
& +0.54005316 \times 10^{-2} \exp(-0.083974802t) \\
& +0.86123492 \times 10^{-2} \exp(-0.010908388t), \\
\\
c(t)_{[5/4]} = {} & 0.097900879 - 0.0014605035 \exp(-0.30159029t) \\
& -0.68930622 \times 10^{-3} \exp(-0.18863972t) \\
& +0.054460116 \exp(-0.092296567t) \\
& +0.56888175 \times 10^{-2} \exp(-0.011874625t), \\
\\
e(t)_{[5/4]} = {} & 0.23775395 + 1.8641738 \times 10^{-8} \exp(-0.58305125t) \\
& +0.21834747 \times 10^{-3} \exp(-0.19187867t) \\
& -.12182142 \exp(-0.092205964t) \\
& +0.017549108 \exp(-0.010961758t).
\end{aligned} \tag{16}
$$

7. Discussion

Figures 1-4 show a comparison between the Fehlberg fourth-fifth order Runge-Kutta method with degree four interpolant (RKF45) [37, 38] solution (built-in function of Maple software) for the dynamics of smoking habit in Spain (3) and the analytic approximations (13), (14), (15) and (16). Moreover, in order to obtain a good numerical reference the accuracy of RKF45 was set to an absolute error of 10^{-7} and relative error of 10^{-6}.

The SM solution (13) is easily obtained by a straightforward procedure. However, as depicted in figures 1-4 and Table 1, the power series solution diverge for large periods of time. Therefore, Padé resummation method was successfully applied to power series (13), resulting in a good agreement with RKF45 results for a period of 200 years.

In order to enlarge the convergence even more, Laplace-Padé resummation method was applied to (13), resulting a good accuracy of the approximation for a period longer than 200 years. Moreover, from Table 1, we can observe a comparison of the exact asymptote (5) of model (3) and the proposed approximations evaluated at $t = \infty$, resulting that LPSM solution is the most accurate approximation in the range of $t = [0, \infty]$ years. To illustrate this, we plot in Figure 5 a comparison between the LPSM solution, the HAM-Padé result [2] and the exact solution for $s(t)$.

Finally, as aforementioned, the resummation methods are a powerful tool to enlarge the domain of convergence of power series solutions that can accurately describe asymptotic non-linear problems like the evolution of the prevalence of the smoking habit in Spain.

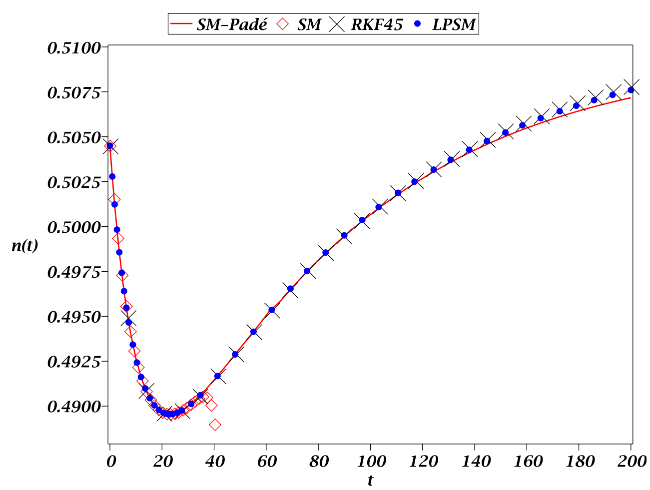

FIGURE 1. Solution of series method with 20 terms for $n(t)$ (diamonds), RKF45 solution (diagonal cross), SM-Padé solution (solid line) and LPSM solution (dots)

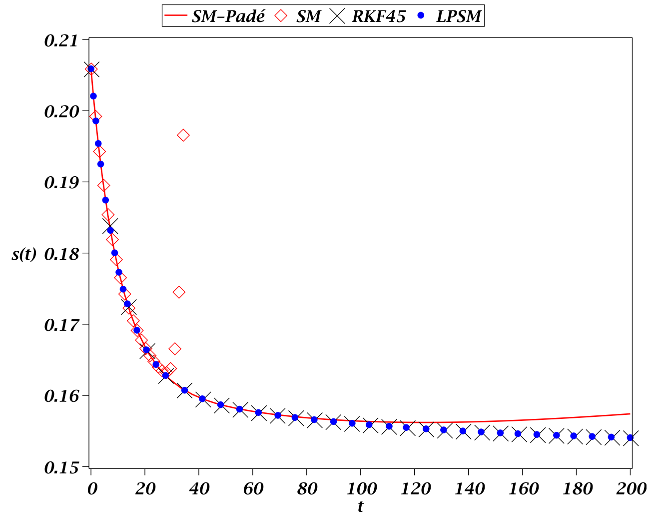

FIGURE 2. Solution of SM method with 20 terms for $s(t)$ (diamonds), RKF45 solution (diagonal cross), SM-Padé solution (solid line) and LPSM solution (dots)

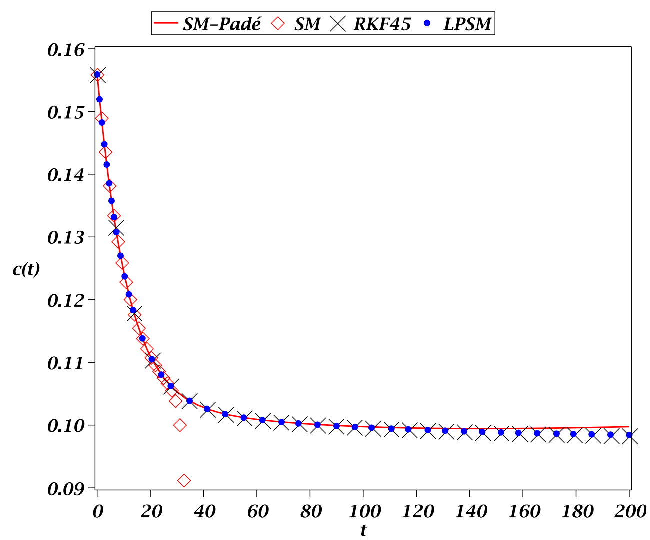

FIGURE 3. Solution of series method with 20 terms for $c(t)$ (diamonds), RKF45 solution (diagonal cross), SM-Padé solution (solid line) and LPSM solution (dots)

Variable	Exact (5)	SM (13)	SM-Padé (14)	LPSM (15), (16)	HAM [2]	HAM-Padé [2]
$n(\infty)$	0.51143607	$-\infty$	0.50450001	0.51085122	$-\infty$	0.51073075
$s(\infty)$	0.15308033	$-\infty$	0.20590000	0.15322657	$+\infty$	0.57420183
$c(\infty)$	0.097648961	$+\infty$	0.15590000	0.097900879	$-\infty$	0.08620481
$e(\infty)$	0.23783464	$-\infty$	0.13370000	0.23775484	$-\infty$	0.24404096

TABLE 1. Exact and approximations of asymptote for model of smoking habit in Spain.

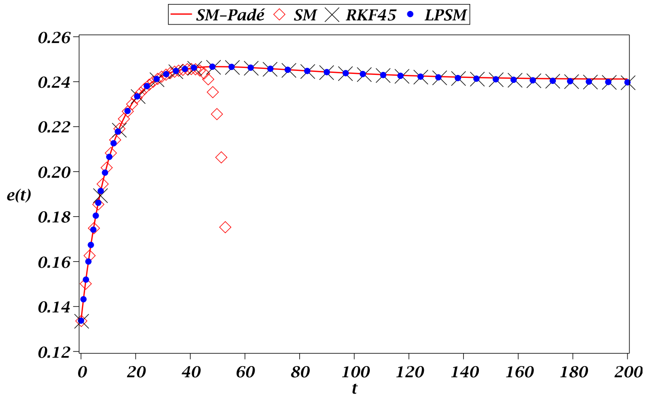

FIGURE 4. Solution of series method with 20 terms for $e(t)$ (diamonds), RKF45 solution (diagonal cross), SM-Padé solution (solid line) and LPSM solution (dots)

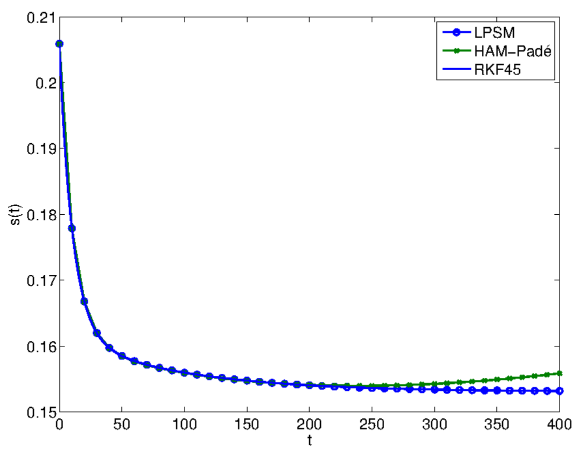

FIGURE 5. Comparison between LPSM solution (circles), HAM-Padé solution (dots) and RKF45 solution (solid line) for $s(t)$.

8. CONCLUDING REMARKS

In this paper, powerful analytical methods SM, SM-Padé and LPSM methods are applied to construct approximated analytical solutions for the model of the evolution of smoking habit in Spain. The series method (SM) provides solutions in the form of fast convergent series with easily computable components. After that, we applied Padé and Laplace-Padé resummation methods to successfully enlarge domain of convergence. The numerical experiments and error analysis are presented to support the theoretical results. Our solutions agree well with the pure numerical solutions and are better than the approximated solution obtained using HAM-Padé approximant in [2]. Finally, the comparison tables also show that LPSM provides the most accurate solution for a period of $t = [0, \infty]$ years.

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