



## A handy exact solution for flow due to a stretching boundary with partial slip

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In this article we provide an exact solution to the nonlinear differential equation that describes the behaviour of a flow due to a stretching flat boundary due to partial slip. For this, we take as a guide the search for an asymptotic solution of the aforementioned equation.

**Keywords:** Nonlinear differential equations; partial slip; stretching boundary; non-Newtonian fluids.

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### 1. Introduction

The flow due to a stretching boundary is important in many engineering processes. For instance in the glass fibre drawing, crystal growing, polymer industries [2], and extrusion processes for the production of plastic sheets [2,3,8,9,10], among many others. Unlike what happens with Newtonian fluids (water, mercury, glycerine, etc.) which usually use no slip boundary conditions, [7] (pages 353-355), there are cases where partial slip between the fluid and the moving surface may occur. Examples include emulsions, as mustard and paints; solutions of solids in liquids finely pulverized, such as the case of clay, and polymer solutions [4].

In these cases the boundary conditions are adequately described by Navier's condition, which states that the amount of relative slip is proportional to local shear stress [4]. As aforementioned, in this study we provide an exact solution to the nonlinear differential equation that describes the behaviour of a flow due to partial slip, and therefore a non Newtonian fluid should be involved. Nevertheless for the sake of simplicity, we will consider the adequate limit cases, in order to justify the use of Newtonian equations for fluids.

For instance, the Newtonian approximation for Bingham plastics (like the clay) amounts to consider fluids, in the limit of small values of the so called, yield stress [1] (page. 233). On the other hand, for the case of pseudo-plastic non Newtonian fluids (generally, aqueous solutions of water soluble polymers show this behaviour), their Newtonian limit amounts to consider the limits for large values of shear rate and apparent viscosity constant [1] (page. 233).

Although there are solutions to this problem [8,9,10], the solving procedures are not easy to follow for undergraduates in physics, mathematics and engineering. Therefore, we propose a straightforward methodology, based on elementary differential and integral calculus, employing as a guide

the search for an asymptotic solution of the afore mentioned problem. The systematic procedures used to determine qualitatively, the asymptotic behaviour for solutions of a differential equation, belong to the qualitative theory of nonlinear differential equations [11] (page. 334). Unlike the above, this study proposes an asymptotic analytical solution, which turns out to be the exact solution of the problem.

### 2. Governing Equations

Consider a two dimensional stretching boundary (see Fig. 1). Where the velocity of the boundary is approximately proportional to the distance  $X$  to the origin [6], so that

$$U = bX. \quad (1)$$

Let  $(u, v)$  be the fluid velocities in the  $(X, Y)$  directions, respectively. In this case, the boundary conditions are adequately described by Navier's condition which states that the amount of relative slip is proportional to local shear stress.

$$u(X, 0) - U = k\nu \frac{\partial u}{\partial Y}(X, 0), \quad (2)$$

where  $k$  is a proportional constant and  $\nu$  is the kinematic viscosity of the bulk fluid.

The relevant equations for this case are Navier-Stokes

$$uu_X + vv_Y + \frac{p_X}{\rho} - \nu(u_{XX} + u_{YY}) = 0, \quad (3)$$

$$uv_X + vv_Y + \frac{p_Y}{\rho} - \nu(v_{XX} + v_{YY}) = 0, \quad (4)$$

and continuity

$$u_X + v_Y = 0, \quad (5)$$

where  $\rho$  and  $p$  are density and pressure, respectively.

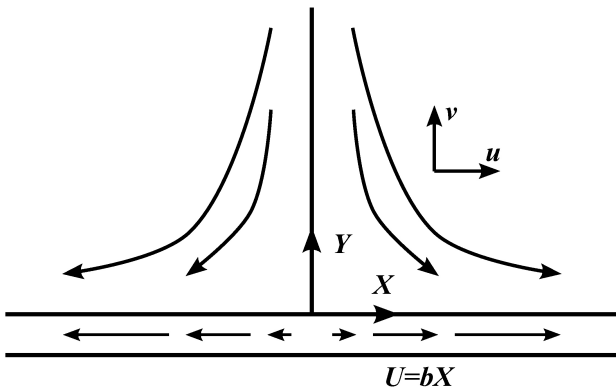


FIGURE 1. Schematic showing a stretching boundary.

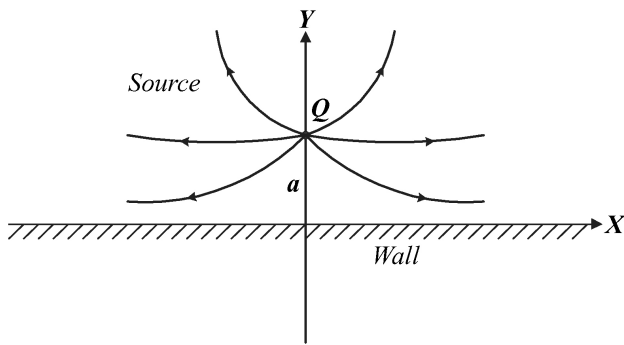


FIGURE 2. Source in front of a wall.

In order to satisfy the equation of incompressibility (5), we will motivate a transformation, which contains implicitly the functional form of the velocity field. We will see that in this fashion, the motion equations are reduced to a single ordinary nonlinear differential equation.

To this end, consider the case of a source of intensity  $Q$  in front of a wall (axis  $X$ ), as it is shown in Fig. 2 [1].

By symmetry arguments, it is expected that streamlines pattern shown in Fig. 2 have, in general terms, a symmetry similar to those in Fig. 1 as  $a$  increases.

From [1] it is possible to show that the value of  $v$  component, when  $a \rightarrow \infty$ , is

$$v(Y) = \frac{-YQ}{2\pi a^2},$$

noticing that  $v < 0$  and it is only a function of  $Y$ .

Using as a guide the above argument, we define a function  $y(x)$ , such that

$$v \approx -y(x), \tag{6}$$

where  $y(x) \geq 0, 0 \leq x < \infty$  and

$$x = Y\sqrt{\frac{b}{\nu}}. \tag{7}$$

From (5) and (6) we deduce that

$$u_X = -v_Y \approx y_Y(x) = \sqrt{\frac{b}{\nu}}y'(x),$$

that is

$$u_X \approx \sqrt{\frac{b}{\nu}}y'(x), \tag{8}$$

where prime denotes differentiation with respect to  $x$ .

Since  $y = y(x)$ , after integrating (8), we obtain

$$u \approx \sqrt{\frac{b}{\nu}}y'(x)X, \tag{9}$$

(by choosing an arbitrary function of  $x$ , zero).

In order to simplify the equations of motion, we introduce the following constant of proportionality into (6)

$$v = -\sqrt{b\nu}y(x), \tag{10}$$

and, therefore, (9) is rewritten as

$$u = by'(x)X. \tag{11}$$

Thus, expressing velocity field according to (10), (11), and (7); (5) is automatically satisfied. Next, we will show that (3) adopts a simpler form under these assumptions.

By using (7), (10), and (11), (3) can be rewritten as

$$uby'(x) + \sqrt{\frac{b}{\nu}}bvXy''(x) - \frac{p_x}{\rho} - b^2Xy'''(x) = 0, \tag{12}$$

where we have employed the chain rule from differential calculus.

It should be noted that  $p_x = 0$ , since the fluid motion is caused by the fluid being dragged along by the moving boundary, therefore

$$uby'(x) + \sqrt{\frac{b}{\nu}}bvXy''(x) - b^2Xy'''(x) = 0, \tag{13}$$

substituting (10) and (11) into (13), we obtain

$$y''' - y'^2 + yy'' = 0. \tag{14}$$

To deduce the boundary conditions of (14), we see that  $v(Y = 0) = 0$ , (see Fig. 1); therefore, from (7) and (10), is clear that

$$y(0) = 0, \tag{15}$$

in the same way, from the condition  $\lim_{Y \rightarrow \infty} u(X, Y) = 0$ , and (7) (see Fig. 1), we obtain the following boundary condition

$$y'(\infty) = 0. \tag{16}$$

To conclude, by substituting (1) and (11) into the Navier's condition (2), we obtain

$$bXy'(0) - bX = k\nu \frac{\partial u}{\partial Y}(X, 0), \tag{17}$$

by the chain rule we rewrite (17) as

$$bXy'(0) - bX = k\nu \left( \frac{\partial u}{\partial x}(X, 0) \right) \left( \frac{dx}{dY} \right), \tag{18}$$

after substituting (7) and (11) we get

$$y'(0) = 1 + Ky''(0), \tag{19}$$

where we have defined

$$K = k\sqrt{b\nu}. \tag{20}$$

In the next section we will solve (14) with boundary conditions (15), (16), and (18).

### 3. The Exact solution of a two dimensional Viscous Flow Equation

In order to obtain an exact solution for (14) we take as a guide the search for an asymptotic solution of the same equation.

Rewriting (14) in the form

$$y = \frac{y'^2 - y'''}{y''}, \tag{21}$$

and defining

$$y_1 = y', \tag{22}$$

it is possible to express (21) as

$$y_1'' - y_1^2 + yy_1' = 0, \tag{23}$$

and the derivative of (21) as follows

$$y_1 = \frac{y_1'(2y_1y_1' - y_1''') - y_1''(y_1^2 - y_1'')}{y_1'^2}. \tag{24}$$

Equations (23) and (24) take the following limit forms when  $y_1(\infty) \rightarrow 0$  (boundary condition (16))

$$y_1'' + Ly_1' = 0, \tag{25}$$

$$y_1''^2 - y_1'y_1''' = 0, \tag{26}$$

(because the condition  $y_1(\infty) \rightarrow 0$  implies that  $y(\infty) \rightarrow L$ , where  $L$  is constant. Also from Fig. 1 and (10), it follows that  $v < 0$  and  $L > 0$ ).

Taking the square of (25) and substituting into (26), we obtain

$$y_1''' = L^2y_1'. \tag{27}$$

In order to solve (27), we propose the following change of variable

$$z = y_1', \tag{28}$$

in such a way that (27) adopts the form

$$z'' = L^2z. \tag{29}$$

Equation (29) has the known solution

$$z(x) = A \exp(Lx) + B \exp(-Lx), \tag{30}$$

where  $A$  and  $B$  are constants.

To avoid that  $z \rightarrow \infty$ , when  $x \rightarrow \infty$  (from (16), (22), and (28) is clear that  $z \rightarrow 0$  for that limit), we choose  $A = 0$  so that (30) adopts the simpler form

$$z(x) = B \exp(-Lx), \tag{31}$$

also, from (28) and (31), we obtain

$$y_1' = B \exp(-Lx), \tag{32}$$

therefore, after integrating the above equation, we obtain

$$y_1 = -\frac{B}{L} \exp(-Lx) + c_1. \tag{33}$$

The condition  $y_1(\infty) = 0$  (see (16) and (22)) leads to  $c_1 = 0$ , therefore

$$y_1 = -\frac{B}{L} \exp(-Lx), \tag{34}$$

after integrating (34), is obtained

$$y(x) = \frac{B}{L^2} \exp(-Lx) + c_2. \tag{35}$$

Since  $y(0) = 0$ , then  $c_2 = -(B/L^2)$ , and (35) becomes

$$y(x) = \frac{B}{L^2} (\exp(-Lx) - 1). \tag{36}$$

Finally, the condition  $y(\infty) \rightarrow L$  is satisfied by choosing  $B = -L^3$ , so that

$$y(x) = L(1 - \exp(-Lx)). \tag{37}$$

On the other hand, from (37), we deduce that

$$y'(0) = L^2, \tag{38}$$

and

$$y''(0) = -L^3. \tag{39}$$

The substitution of (38) and (39) into (19) leads to a general relation between  $K$  and the asymptotic form of the solution given by  $y = L$

$$KL^3 + L^2 = 1. \tag{40}$$

For the case  $K = 0$  [4,5], (40) and (37) adopt the form

$$L = 1, \tag{41}$$

$$y(x) = 1 - \exp(-x), \quad 0 \leq x \leq \infty, \tag{42}$$

respectively.

### 4. Discussion

The substitution of (37) into (14) reveals that (37) is, indeed, an exact general solution not just an approximation; although our process was aimed to find an asymptotic solution that would satisfy boundary conditions (15), (16), and (19). As a matter of fact, for the particular case of inviscid flow  $K = 0$  ((41) and (42)) was reported in [4,5] as a rare closed exact solution for (14).

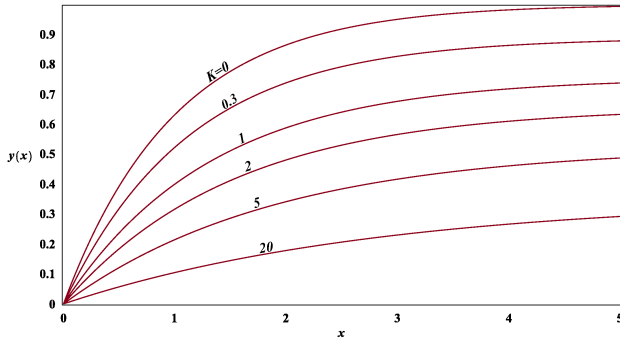


FIGURE 3. Function  $y(x)$  for several values of  $K$ .

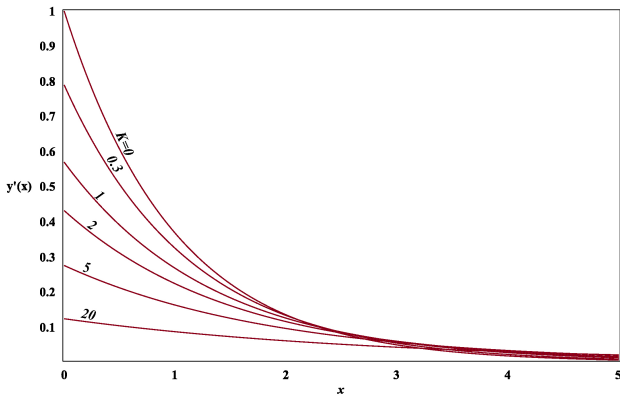


FIGURE 4. Function  $y'(x)$  for several values of  $K$ .

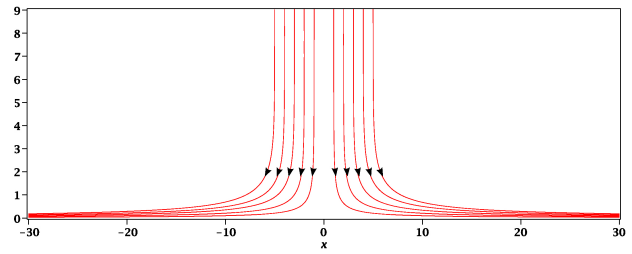


FIGURE 5. Streamlines for  $K = 0$ .

It is noteworthy that (40) relates the asymptotic form of the solution  $y = L$  with the constant  $K$ , and the latter is related to the fluid viscosity (see (20)); so that, in principle, (40) determines in advance the asymptotic value of the solution, from the value of the viscosity. Similarly, (39) and (40) provide a general way to determine the value of  $y''(0)$  in terms of  $K$ . These values are often difficult to calculate and in the literature can be found tables that provide some values but just for a few values of  $K$  [4].

Figure 3 and Fig. 4 show functions  $y(x)$  and  $y'(x)$  respectively for various values of  $K$ ; these functions determine the velocity field through (10) and (11). Figure 5 shows a sketch for several streamlines when  $K = 0$ .

### 5. Conclusion

An important task is to find analytic expressions that provide a good description of the solution to the nonlinear differential equations like (14). For instance, the flow induced by a stretching sheet is adequately described by (37) and (40). An important result for practical applications it follows that (40) relates the asymptotic form of the solution  $y = L$ , with the fluid viscosity, so that in principle (40) determines in advance the asymptotic value of the solution from the value of the viscosity.

This work showed, by means of a simple procedure, that some nonlinear differential equations may be solved in exact form taking as a guide the search for an asymptotic solution of the same equation. Is clear that this procedure could be useful, at least, to find approximate solutions for some equations.

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