

Homogeneously polyanalytic kernels on the unit ball and the Siegel domain

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CINVESTAV, México

International Workshop Operator Theory and Function Spaces
Universidad Veracruzana, September 2022

Outline

§ Pushforward reproducing kernel

§ Polyanalytic functions

§ Mean value property

§ The poly-Bergman kernel

A brief history of the Poly-Bergman space

§ Koshelov (1977)

- f is m -analytic in $\mathbb{D}$, then $f(0) = \int f(z)P(|z|^2)dz$.
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§ Hachadi & Youssfi (2019)

- Considered radial measures in polyanalytic spaces over $\mathbb{D}$ and $\mathbb{C}$.
- Computed RK of $\mathcal{A}_m^2(\mathbb{D}, \mu_\alpha)$.

Reproducing Kernel Hilbert Spaces

Def. Let X be a set and H be a Hilbert space, $H \leq \mathbb{C}^X$.

A family of functions $(K_x)_{x \in X} \in H^X$ is said to be a RK for H if

$$\forall f \in H, \quad \forall x \in X, \quad f(x) = \langle f, K_x \rangle.$$

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Some properties:

- If there exists a RK then it is unique.
- $K_x(y) = \langle K_x, K_y \rangle$.
- $K_x(y) = \overline{K_y(x)}$.
- $\text{span}(\{K_x : x \in X\})$ is dense in H .

More properties

If $(b_j)_{j \in \mathbb{N}}$ is an orthonormal basis for H ,

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For $x \in X$,

$$\text{eval}_x: H \rightarrow \mathbb{C}, \quad \text{eval}_x(f) := f(x).$$

Then are equivalent:

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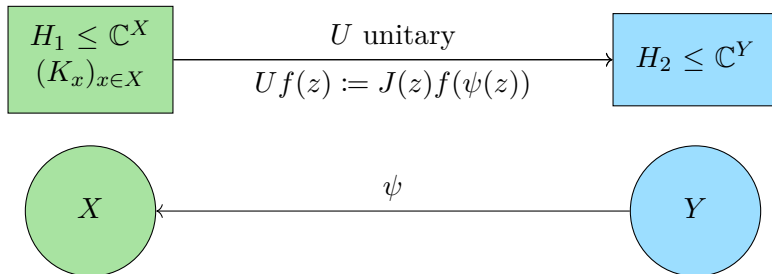
Examples:

$$\mathcal{A}^2(\mathbb{D}) : K_z^{\mathbb{D}}(w) = \frac{1}{(1 - \bar{z}w)^2},$$

$$\mathcal{F}^2(\mathbb{C}) : K_z^{\mathbb{C}}(w) = e^{w\bar{z}}.$$

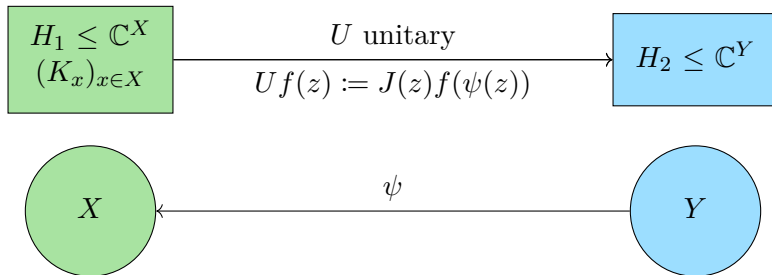
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Then the RK for H_2 is

$$K_z^{H_2}(w) = \overline{J(z)} J(w) K_{\psi(z)}^{H_1}(\psi(w)).$$

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$$g(z) = (Uf)(z)$$

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Then

$$g(z) = (Uf)(z) = J(z)f(\psi(z)) = J(z) \langle f, K_{\psi(z)}^{H_1} \rangle = \langle g, \overline{J(z)} U K_{\psi(z)}^{H_1} \rangle,$$

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which means that $K_z^{H_2} = \overline{J(z)} U K_{\psi(z)}^{H_1}$.

The Wirtinger derivative

One-dimensional case:

$$\frac{\partial}{\partial \bar{z}_j} := \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).$$

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For n -dimensional case, $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}_0^n$,

$$\bar{D}^{\mathbf{k}} := \frac{\partial^{|\mathbf{k}|}}{\partial^{k_1} \bar{z}_1 \dots \partial^{k_n} \bar{z}_n}.$$

$$|\mathbf{k}| = \sum_{s=1}^n k_s,$$

$$\mathbf{k}! = \prod_{s=1}^n k_s!,$$

$$z^{\mathbf{k}} = \prod_{s=1}^n z_s^{k_s}.$$

Polyanalytic functions

Let Ω be an open set in $\mathbb{C}^n$.

Def. A function f is said to be $\mathbf{k}$ -analytic if $\overline{D}^{\mathbf{k}} f = 0$, i.e.,

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Def. Let $m \in \mathbb{N}$. A function f is said to be *homogeneously polyanalytic of order m* (or simply m -analytic) if

$$\overline{D}^{\mathbf{k}} f = 0, \quad \forall \mathbf{k} \in \mathbb{N}_0^n \text{ with } |\mathbf{k}| = m.$$

This class is defined by $\binom{n+m-1}{m}$ equations.

Polyanalytic functions

$$\mathcal{A}_m(\Omega) := \left\{ f \in C^m(\Omega) : \quad \forall \mathbf{k} \in \mathbb{N}_0^n \quad (|\mathbf{k}| = m \quad \Rightarrow \quad \overline{D}^{\mathbf{k}} f = 0) \right\}.$$

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$$\tilde{\mathcal{A}}_m(\Omega) := \left\{ \sum_{|\mathbf{k}| < m} h_{\mathbf{k}}(z) \overline{z}^{\mathbf{k}} : \quad h_{\mathbf{k}} \in \mathcal{A}(\Omega) \right\}.$$

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Theorem.

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Theorem.

$$\mathcal{A}_m(\Omega) = \tilde{\mathcal{A}}_m(\Omega).$$

Example: A polyanalytic function of order 2: $f(z) = 1 - |z|^2$. Then

$$f|_{\mathbb{S}_n} \equiv 0, \quad \text{while} \quad f \neq 0.$$

Polyanalytic functions as power series

$$(\forall \mathbf{k}, |\mathbf{k}| = m) \quad \overline{D}^{\mathbf{k}} f = 0 \quad \Longleftrightarrow \quad f(z) = \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^n \\ |\mathbf{k}| < \textcolor{red}{m}}} h_{\mathbf{k}}(z) \overline{z}^{\mathbf{k}}.$$

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Cor. If $f \in \mathcal{A}_m(\mathbb{B}^n)$, then

$$f(z) = \sum_{|\mathbf{k}| < m} h_{\mathbf{k}}(z) \bar{z}^{\mathbf{k}} = \sum_{|\mathbf{k}| < m} \left(\sum_{\mathbf{j} \in \mathbb{N}_0^n} \beta_{\mathbf{j}, \mathbf{k}} z^{\mathbf{j}} \right) \bar{z}^{\mathbf{k}},$$

where the series converges uniformly on compact subsets of $\mathbb{B}^n$.

Invariance

Prop. Let $M \in \mathbb{C}^{n \times n}$ invertible and $f \in \mathcal{A}_m(\Omega)$. Define

$$g(z) = f(M^{-1}z).$$

Then $g \in \mathcal{A}_m(M\Omega)$.

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Example: Let $\mathbf{k} = (2, 1) \in \mathbb{N}_0^2$. The space $\mathcal{A}_{(2,1)}(\mathbb{C}^2)$ is not invariant under linear change of variables:

$$f(z_1, z_2) = \overline{z_1}^3,$$

$$g(z_1, z_2) = f(z_1 + z_2, z_1 - z_2) = (z_1 + z_2)^3 = \overline{z_1}^3 + 3\overline{z_1}^2\overline{z_2} + 3\overline{z_1}\overline{z_2}^2 + \overline{z_2}^3.$$

$$f \in \mathcal{A}_{(2,1)}(\mathbb{C}^2) \quad \text{but} \quad g \notin \mathcal{A}_{(2,1)}(\mathbb{C}^2).$$

Jacobi polynomials

By Rodrigues formula:

$$P_m^{(\alpha,\beta)}(x) = \frac{(-1)^m}{2^m m!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^m}{dx^m} \left[(1-x)^{m+\alpha} (1+x)^{m+\beta} \right].$$

If $\alpha, \beta > -1$, then $(P_m^{(\alpha,\beta)})_{m \in \mathbb{N}_0}$ is orthogonal in $(-1, 1)$ with respect to the weight

$$(1-x)^\alpha (1+x)^\beta.$$

The key polynomial

Define

$$R_m^{(\alpha, \beta)}(t) := \frac{(-1)^m B(\alpha + 1)(\beta + 1)}{B(\alpha + m + 1, \beta + 1)} P_m^{(\alpha, \beta + 1)}(2t - 1).$$

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Prop. For every polynomial h of $\deg(h) \leq m - 1$,

$$h(0) = \frac{1}{B(\alpha + 1, \beta + 1)} \int_0^1 h(t) R_m^{(\alpha, \beta)}(t) (1 - t)^\alpha t^\beta dt.$$

Integrals over $\mathbb{B}_n$

For an integrable function f on $\mathbb{B}^n$,

$$\int_{\mathbb{B}_n} f \, d\mu = \int_0^1 r^{2n-1} \left(\int_{\mathbb{S}_n} f(r\zeta) \, d\mu_{\mathbb{S}_n}(\zeta) \right) dr.$$

Also, for the elements on the unit sphere,

$$\int_{\mathbb{S}_n} \zeta^{\mathbf{j}} \bar{\zeta}^{\mathbf{k}} \, d\mu_{\mathbb{S}_n}(\zeta) = \frac{2\pi^n \mathbf{j}!}{(n-1+|\mathbf{j}|)!} \cdot \delta_{\mathbf{j},\mathbf{k}}.$$

The reproducing property of $R_{m-1}^{(\alpha, n-1)}$

Prop. Let $f \in \mathcal{A}_m(\mathbb{B}_n) \cap L^2(\mathbb{B}_n, d\mu_\alpha)$. Then

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Proof. Put $z = r\zeta$, with $0 \leq r < 1$ and $\zeta \in \mathbb{S}_n$. Then

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$$\int_{\substack{\xi \mathbb{B}_n \\ \xi < 1}} f(z) R_{m-1}^{(\alpha, n-1)}(|z|^2) d\mu_\alpha z = \sum_{|\mathbf{k}| < m} \beta_{\mathbf{k}, \mathbf{k}} \cdot (\text{coef})_{\mathbf{k}} \int_0^\xi R_{m-1}^{(\alpha, n-1)}(t) (1-t)^\alpha t^{n-1+|\mathbf{k}|} dt.$$

Taking limits when $\xi \rightarrow 1^-$ and Lebesgue's Dom. Conv. Thm.

Weighted poly-Bergman spaces

Let Ω be an open set of $\mathbb{C}^n$, $W \in C(\Omega, (0, +\infty))$, and $m \in \mathbb{N}_0$,
Provide Ω with a measure $d\nu = W d\mu_{\text{Leb}}$.

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$\mathcal{A}_m^2(\Omega, \nu)$ is a **closed subspace** of $L^2(\Omega, \nu)$.

For any compact subset K of Ω there exists $C_{m,W,K} > 0$ s.t.

$$|f(z)| \leq C_{m,W,K} \|f\|_{L^2(\Omega, d\nu)} \quad \left(f \in \mathcal{A}_m^2(\Omega, \nu), z \in K \right).$$

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$$\mathcal{A}_{(m)}^2(\Omega, d\nu) := \mathcal{A}_m^2(\Omega, d\nu) \ominus \mathcal{A}_{m-1}^2(\Omega, d\nu)$$

The true poly-Bergman space of order m .

Poly-Bergman space over $\mathbb{B}_n$

$$d\mu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d$$

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Main goal: Compute the RK of $\mathcal{A}_m^2(\mathbb{B}_n, \mu_\alpha)$.

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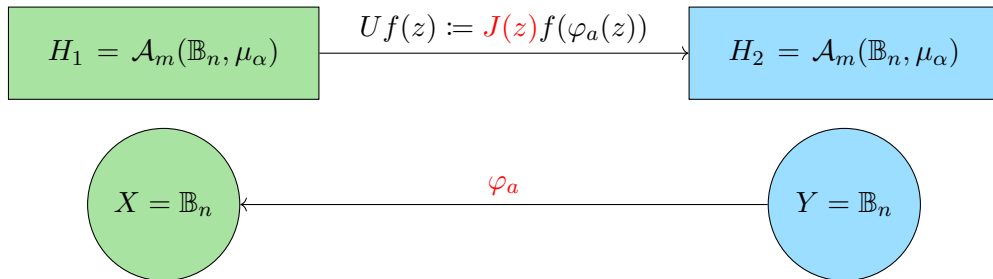
$$f(\mathbf{0}) = \int_{\mathbb{B}_n} f(w) R_{m-1}^{(\alpha, n-1)}(|w|^2) d\mu_\alpha(w).$$

This means that

$$K_{\mathbf{0}}^{\mathbb{B}_n}(w) = R_{m-1}^{(\alpha, n-1)}(|w|^2).$$

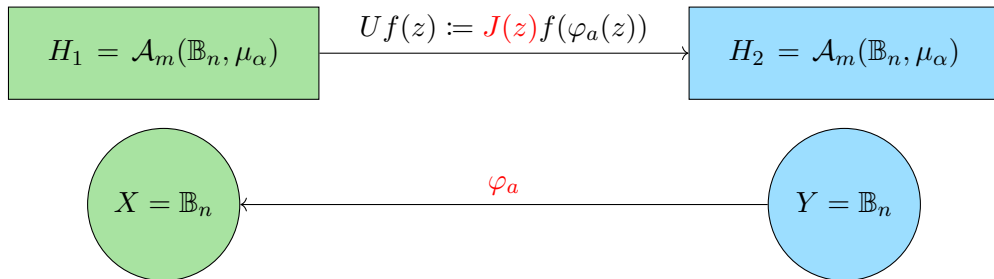
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Moving the kernel

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- $\varphi_a: \mathbb{B}_n \rightarrow \mathbb{B}_n$, s.t. $\varphi_a(a) = 0$,
- U is a unitary operator.

Building the operator U

Consider

- The Möbius transform

$$\varphi_a(z) = \frac{a - \frac{\langle z, a \rangle}{\langle a, a \rangle} a - \sqrt{1 - |a|^2} \left(z - \frac{\langle z, a \rangle}{\langle a, a \rangle} a \right)}{1 - \langle z, a \rangle},$$

which interchanges any point $z \in \mathbb{B}_n$ with a .

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- The *Pessoa's factor*

$$p_{m,a}(z) = \left(\frac{1 - \langle a, z \rangle}{1 - \langle z, a \rangle} \right)^{m-1},$$

which preserves polyanalyticity:

$$f \in \mathcal{A}_m(\mathbb{B}_n) \quad \implies \quad p_{m,a} \cdot (f \circ \varphi_a) \in \mathcal{A}_m(\mathbb{B}_n).$$

Building the operator U

- The *isometry factor*

$$g_{\alpha,a}(z) = \left(\frac{1 - |a|^2}{(1 - \langle z, a \rangle)^2} \right)^{\frac{\alpha+n+1}{2}},$$

which preserves the norm:

$$f \in L^2(\mathbb{B}_n, \mu_\alpha) \quad \implies \quad \|g_{\alpha,a} \cdot (f \circ \varphi_a)\|_{L^2(\mathbb{B}_n, \mu_\alpha)} = \|f\|_{L^2(\mathbb{B}_n, \mu_\alpha)}.$$

The reproducing kernel over the unit ball

Theorem. The RK of the space $\mathcal{A}_m^2(\mathbb{B}_n, \mu_\alpha)$ is given by

$$K_z^{\mathbb{B}_n}(w) = \frac{(1 - \langle z, w \rangle)^{m-1}}{(1 - \langle w, z \rangle)^{n+m+\alpha}} R_{m-1}^{(\alpha, n-1)}(|\varphi_z(w)|^2).$$

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Proof. Use the unitary operator

$$(Uf)(w) := p_{m,z}(w)g_{\alpha,z}(w)f(\varphi_z(w)).$$

The Siegel domain

$$\mathbb{H}_n := \left\{ \xi = (\xi', \xi_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \operatorname{Im}(\xi_n) - |\xi'|^2 > 0 \right\},$$
$$d\nu_\alpha := \frac{c_\alpha}{4} \left(\operatorname{Im}(\xi_n) - |\xi'|^2 \right)^\alpha d\mu_{\mathbb{H}_n}(\xi).$$

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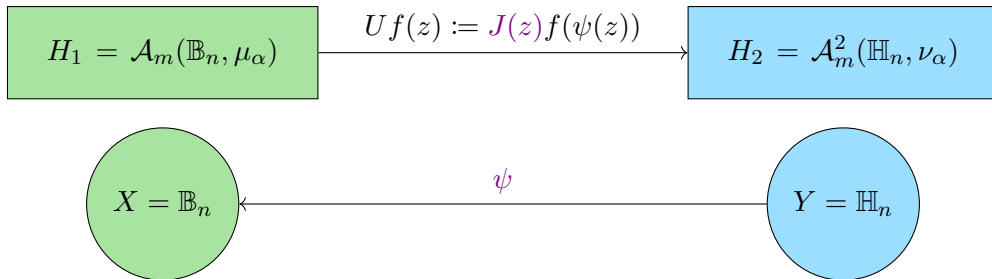
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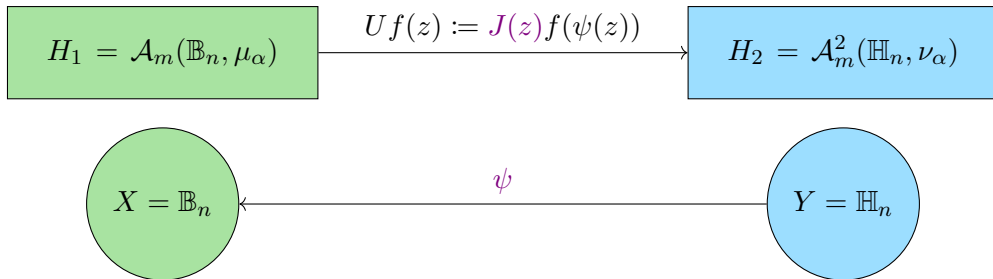
Pseudohyperbolic metric:

$$\rho_{\mathbb{B}_n}(z, w) := |\varphi_z(w)|, \quad \rho_{\mathbb{H}_n}(\xi, \eta) := \rho_{\mathbb{B}_n}(\psi(\xi), \psi(\eta)).$$

The reproducing kernel over the Siegel domain



The reproducing kernel over the Siegel domain



$$K_{\xi}^{\mathbb{H}_n}(\eta) = \frac{\left(\frac{\xi_n - \overline{\eta_n}}{2i} - \langle \xi', \eta' \rangle\right)^{m-1}}{\left(\frac{\eta_n - \overline{\xi_n}}{2i} - \langle \eta', \xi' \rangle\right)^{n+m+\alpha}} R_{m-1}^{(\alpha, n-1)} \left(\rho_{\mathbb{H}_n}(\xi, \eta)^2\right).$$

Bibliography



Maximenko, E.; Ramos-Vazquez, G.; L-P (2021):

Homogeneously polyanalytic kernels on the unit ball and the Siegel domain.

<https://doi.org/10.1007/s11785-021-01145-z>

Bibliography



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A simultaneous and independent work by Youssfi:

Polyanalytic reproducing kernels in $\mathbb{C}^n$.

<https://hal.archives-ouvertes.fr/hal-03131190>

RKHS
○○○○

Polyanalytic functions
○○○○○

Mean value property
○○○○○

Poly-Bergman kernel
●○○○○○○○

Thank you.