Homogeneously polyanalytic kernels on the unit ball and the Siegel domain

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Outline

§ Pushforward reproducing kernel

§ Polyanalytic functions

§ Mean value property

§ The poly-Bergman kernel

A brief history of the Poly-Bergman space

§ Koshelov (1977)

- f is m-analytic in $\mathbb{D}$, then $f(0) = \int f(z)P(|z|^2)\mathrm{d}z$.
- Computed the RK of $\mathcal{A}_m^2(\mathbb{D})$ using Möbius transformations.

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§ Pessoa (2013)

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- Proposed a correcting factor to restore the polyanalyticity.
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§ Hachadi & Youssfi (2019)

- Considered radial measures in polyanalytic spaces over $\mathbb{D}$ and $\mathbb{C}$.
- Computed RK of $\mathcal{A}_m^2(\mathbb{D}, \mu_\alpha)$.

Reproducing Kernel Hilbert Spaces

Def. Let X be a set and H be a Hilbert space, $H \leq \mathbb{C}^X$.
A family of functions $(K_x)_{x\in X} \in H^X$ is said to be a RK for H if

$$\forall f \in H, \qquad \forall x \in X, \qquad f(x) = \langle f, K_x \rangle.$$

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Some properties:

- If there exists a RK then it is unique.
- $K_x(y) = \langle K_x, K_y \rangle$.
- $K_x(y) = \overline{K_y(x)}$.
- $\operatorname{span}(\{K_x : x \in X\})$ is dense in H.

More properties

If $(b_j)_{j\in\mathbb{N}}$ is an othonormal basis for H,

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For $x \in X$,

$$\mathrm{eval}_x : H \to \mathbb{C}, \qquad \mathrm{eval}_x(f) := f(x).$$

Then are equivalent:

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Examples:

$$\mathcal{A}^2(\mathbb{D}) : \; K_z^{\mathbb{D}}(w) = \frac{1}{(1-\overline{z}w)^2}, \qquad \mathcal{F}^2(\mathbb{C}) : \; K_z^{\mathbb{C}}(w) = \mathrm{e}^{w\overline{z}}.$$

Pushforward reproducing kernel

Let H_1 and H_2 be RKHS's.

$$
\begin{tikzcd}
\boxed{\begin{array}{c} H_1 \leq \mathbb{C}^X \\ (K_x)_{x \in X} \end{array}} \arrow[rrr, "U \text{ unitary}", "Uf(z) := J(z)f(\psi(z))"'] & & & \boxed{H_2 \leq \mathbb{C}^Y} \\
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Then the RK for H_2 is

$$K_z^{H_2}(w) = \overline{J(z)}\, J(w)\, K_{\psi(z)}^{H_1}(\psi(w)).$$

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which means that $K_z^{H_2} = \overline{J(z)} U K_{\psi(z)}^{H_1}$.

The Wirtinger derivative

One-dimensional case:

$$\frac{\partial}{\partial \overline{z_j}} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \mathrm{i}\,\frac{\partial}{\partial y_j}\right).$$

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For n-dimensional case, $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$,

$$\overline{D}^{\mathbf{k}} := \frac{\partial^{|\mathbf{k}|}}{\partial^{k_1}\overline{z_1}\cdots\partial^{k_n}\overline{z_n}}.$$

$$|\mathbf{k}| = \sum_{s=1}^{n} k_s, \qquad \mathbf{k}! = \prod_{s=1}^{n} k_s!, \qquad z^{\mathbf{k}} = \prod_{s=1}^{n} z_s^{k_s}.$$

Polyanalytic functions

Let Ω be an open set in $\mathbb{C}^n$.

Def. A function f is said to be $\mathbf{k}$-analytic if $\overline{D}^{\mathbf{k}} f = 0$, i.e.,

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Def. Let $m \in \mathbb{N}$. A function f is said to be *homogeneously polyanalytic of order* m (or simply m-analytic) if

$$\overline{D}^{\mathbf{k}} f = 0, \qquad \forall\, \mathbf{k} \in \mathbb{N}_0^n \text{ with } |\mathbf{k}| = m.$$

This class is defined by $\binom{n+m-1}{m}$ equations.

Polyanalytic functions

$$\mathcal{A}_m(\Omega) := \left\{ f \in C^m(\Omega) \colon \quad \forall\, \mathbf{k} \in \mathbb{N}_0^n \quad (|\mathbf{k}| = m \quad \Rightarrow \quad \overline{D}^{\mathbf{k}} f = 0) \right\}.$$

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$$\widetilde{\mathcal{A}}_m(\Omega) := \left\{ \sum_{|\mathbf{k}|<m} h_{\mathbf{k}}(z) \overline{z}^{\mathbf{k}} : \quad h_{\mathbf{k}} \in \mathcal{A}(\Omega) \right\}.$$

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Theorem.

$$\mathcal{A}_m(\Omega) = \widetilde{\mathcal{A}}_m(\Omega).$$

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Theorem.

$$\mathcal{A}_m(\Omega) = \widetilde{\mathcal{A}}_m(\Omega).$$

Example: A polyanalytic function of order 2: $f(z) = 1 - |z|^2$. Then

$$f|_{\mathbb{S}_n} \equiv 0, \qquad \text{while} \qquad f \neq 0.$$

Polyanalytic functions as power series

$$(\forall\, \mathbf{k}\,,\, |\mathbf{k}| = m) \quad \overline{D}^{\mathbf{k}} f = 0 \qquad \Longleftrightarrow \qquad f(z) = \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^n \\ |\mathbf{k}|<m}} h_{\mathbf{k}}(z)\, \overline{z}^{\mathbf{k}}.$$

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Cor. If $f \in \mathcal{A}_m(\mathbb{B}^n)$, then

$$f(z) = \sum_{|\mathbf{k}|<m} h_{\mathbf{k}}(z)\, \overline{z}^{\mathbf{k}} = \sum_{|\mathbf{k}|<m} \left(\sum_{\mathbf{j}\in\mathbb{N}_0^n} \beta_{\mathbf{j},\mathbf{k}}\, z^{\mathbf{j}} \right) \overline{z}^{\mathbf{k}},$$

where the series converges uniformly on compact subsets of $\mathbb{B}^n$.

Invariance

Prop. Let $M \in \mathbb{C}^{n\times n}$ invertible and $f \in \mathcal{A}_m(\Omega)$. Define

$$g(z) = f(M^{-1}z).$$

Then $g \in \mathcal{A}_m(M\Omega)$.

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Example: Let $\mathbf{k} = (2,1) \in \mathbb{N}_0^2$. The space $\mathcal{A}_{(2,1)}(\mathbb{C}^2)$ is not invariant under linear change of variables:

$$f(z_1, z_2) = \overline{z_1}^3,$$

$$g(z_1, z_2) = f(z_1 + z_2, z_1 - z_2) = \overline{(z_1 + z_2)}^3 = \overline{z_1}^3 + 3\overline{z_1}^2\overline{z_2} + 3\overline{z_1}\,\overline{z_2}^2 + \overline{z_2}^3.$$

$$f \in \mathcal{A}_{(2,1)}(\mathbb{C}^2) \qquad \text{but} \qquad g \notin \mathcal{A}_{(2,1)}(\mathbb{C}^2).$$

Jacobi polynomials

By Rodrigues formula:

$$P_m^{(\alpha,\beta)}(x) = \frac{(-1)^m}{2^m m!}(1-x)^{-\alpha}(1+x)^{-\beta} \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[(1-x)^{m+\alpha}(1+x)^{m+\beta}\right].$$

If $\alpha, \beta > -1$, then $(P_m^{(\alpha,\beta)})_{m\in\mathbb{N}_0}$ is orthogonal in $(-1,1)$ with respect to the weight

$$(1-x)^\alpha(1+x)^\beta.$$

The key polynomial

Define

$$R_m^{(\alpha,\beta)}(t) := \frac{(-1)^m\,\mathrm{B}(\alpha+1)(\beta+1)}{\mathrm{B}(\alpha+m+1,\beta+1)} P_m^{(\alpha,\beta+1)}(2t-1).$$

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Prop. For every polynomial h of $\deg(h) \le m-1$,

$$h(0) = \frac{1}{\mathrm{B}(\alpha+1,\beta+1)} \int_0^1 h(t)\, R_m^{(\alpha,\beta)}(t)\, (1-t)^{\alpha} t^{\beta}\, \mathrm{d}t.$$

Integrals over $\mathbb{B}_n$

For an integrable function f on $\mathbb{B}^n$,

$$\int_{\mathbb{B}_n} f \, \mathrm{d}\mu = \int_0^1 r^{2n-1} \left(\int_{\mathbb{S}_n} f(r\zeta) \, \mathrm{d}\mu_{\mathbb{S}_n}(\zeta) \right) \mathrm{d}r.$$

Also, for the elements on the unit sphere,

$$\int_{\mathbb{S}_n} \zeta^{\mathbf{j}} \, \overline{\zeta}^{\mathbf{k}} \mathrm{d}\mu_{\mathbb{S}_n}(\zeta) = \frac{2\pi^n \mathbf{j}!}{(n-1+|\mathbf{j}|)!} \cdot \delta_{\mathbf{j},\mathbf{k}}.$$

The reproducing property of $R_{m-1}^{(\alpha,n-1)}$

Prop. Let $f \in \mathcal{A}_m(\mathbb{B}_n) \cap L^2(\mathbb{B}_n, \mathrm{d}\mu_\alpha)$. Then

$$f(0) = \int_{\mathbb{B}_n} f(z) R_{m-1}^{(\alpha,n-1)}(|z|^2) \mathrm{d}\mu_\alpha(z).$$

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Proof. Put $z = r\zeta$, with $0 \leq r < 1$ and $\zeta \in \mathbb{S}_n$. Then

$$f(z) = \sum_{|\mathbf{k}|<m} h_{\mathbf{k}}(z)\, \overline{z}^{\mathbf{k}} = \sum_{|\mathbf{k}|<m} \left(\sum_{\mathbf{j} \in \mathbb{N}_0^n} \beta_{\mathbf{j},\mathbf{k}}\, z^{\mathbf{j}} \right) \overline{z}^{\mathbf{k}}$$

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$$\int_{\substack{\xi\mathbb{B}_n \\ \xi<1}} f(z) R_{m-1}^{(\alpha,n-1)}(|z|^2)\mathrm{d}\mu_\alpha z = \sum_{|\mathbf{k}|<m} \beta_{\mathbf{k},\mathbf{k}} \cdot (\mathrm{coef})_{\mathbf{k}} \int_0^{\xi} R_{m-1}^{(\alpha,n-1)}(t)(1-t)^{\alpha} t^{n-1+|\mathbf{k}|}\mathrm{d}t.$$

Taking limits when $\xi \to 1^-$ and Lebesgue's Dom. Conv. Thm.

Weighted poly-Bergman spaces

Let Ω be an open set of $\mathbb{C}^n$, $W \in C(\Omega, (0, +\infty))$, and $m \in \mathbb{N}_0$, Provide Ω with a measure $\mathrm{d}\nu = W\,\mathrm{d}\mu_{\mathrm{Leb}}$.

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$\mathcal{A}^2_m(\Omega, \nu)$ is a **closed subspace** of $L^2(\Omega, \nu)$.

For any compact subset K of Ω there exists $C_{m,W,K} > 0$ s.t.

$$|f(z)| \leq C_{m,W,K} \|f\|_{L^2(\Omega, \mathrm{d}\nu)} \qquad \left(f \in \mathcal{A}^2_m(\Omega, \nu), z \in K\right).$$

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$$\mathcal{A}^2_{(m)}(\Omega, \mathrm{d}\nu) := \mathcal{A}^2_m(\Omega, \mathrm{d}\nu) \ominus \mathcal{A}^2_{m-1}(\Omega, \mathrm{d}\nu)$$

The true poly-Bergman space of order m.

Poly-Bergman space over $\mathbb{B}_n$

$$\mathrm{d}\mu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha \mathrm{d}$$

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Main goal: Compute the RK of $\mathcal{A}_m^2(\mathbb{B}_n, \mu_\alpha)$.

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We have proven the weighted mean value property:

$$f(0) = \int_{\mathbb{B}_n} f(w) R^{(\alpha,n-1)}_{m-1}(|w|^2)\, d\mu_\alpha(w).$$

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This means that

$$K^{\mathbb{B}_n}_0(w) = R^{(\alpha,n-1)}_{m-1}(|w|^2).$$

Moving the kernel

Let $a \in \mathbb{B}_n$.

$$\begin{tikzcd}
H_1 = \mathcal{A}_m(\mathbb{B}_n, \mu_\alpha) \arrow[rr, "Uf(z) := J(z) f(\varphi_a(z))"] & & H_2 = \mathcal{A}_m(\mathbb{B}_n, \mu_\alpha) \\
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- $\varphi_a \colon \mathbb{B}_n \to \mathbb{B}_n$, s.t. $\varphi_a(a) = 0$,
- U is a unitary operator.

Building the operator U

Consider

- The Möbius transform

$$\varphi_a(z) = \frac{a - \frac{\langle z,a\rangle}{\langle a,a\rangle}a - \sqrt{1-|a|^2}\left(z - \frac{\langle z,a\rangle}{\langle a,a\rangle}a\right)}{1-\langle z,a\rangle},$$

which interchanges any point $z \in \mathbb{B}_n$ with a.

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- The Möbius transform

$$\varphi_a(z) = \frac{a - \frac{\langle z,a\rangle}{\langle a,a\rangle}a - \sqrt{1-|a|^2}\left(z - \frac{\langle z,a\rangle}{\langle a,a\rangle}a\right)}{1-\langle z,a\rangle},$$

  which interchanges any point $z \in \mathbb{B}_n$ with a.

- The *Pessoa's factor*

$$p_{m,a}(z) = \left(\frac{1-\langle a,z\rangle}{1-\langle z,a\rangle}\right)^{m-1},$$

  which preserves polyanalyticity:

$$f \in \mathcal{A}_m(\mathbb{B}_n) \quad \implies \quad p_{m,a}\cdot(f\circ\varphi_a) \in \mathcal{A}_m(\mathbb{B}_n).$$

Building the operator U

- The *isometry factor*

$$g_{\alpha,a}(z) = \left(\frac{1-|a|^2}{(1-\langle z,a\rangle)^2}\right)^{\frac{\alpha+n+1}{2}},$$

which preserves the norm:

$$f \in L^2(\mathbb{B}_n,\mu_\alpha) \qquad \Longrightarrow \qquad \|g_{\alpha,a}\cdot(f\circ\varphi_a)\|_{L^2(\mathbb{B}_n,\mu_\alpha)} = \|f\|_{L^2(\mathbb{B}_n,\mu_\alpha)}.$$

The reproducing kernel over the unit ball

Theorem. The RK of the space $\mathcal{A}^2_m(\mathbb{B}_n, \mu_\alpha)$ is given by

$$K_z^{\mathbb{B}_n}(w) = \frac{(1 - \langle z, w\rangle)^{m-1}}{(1 - \langle w, z\rangle)^{n+m+\alpha}} \, R_{m-1}^{(\alpha, n-1)}(|\varphi_z(w)|^2).$$

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Proof. Use the unitary operator

$$(Uf)(w) := p_{m,z}(w) g_{\alpha,z}(w) f(\varphi_z(w)).$$

The Siegel domain

$$\mathbb{H}_n := \left\{\xi = (\xi', \xi_n) \in \mathbb{C}^{n-1} \times \mathbb{C} \colon \operatorname{Im}(\xi_n) - |\xi'| > 0\right\},$$

$$\mathrm{d}\nu_\alpha := \frac{c_\alpha}{4} \left(\operatorname{Im}(\xi_n) - |\xi'|^2\right)^\alpha \mathrm{d}\mu_{\mathbb{H}_n}(\xi).$$

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Cayley transform $\psi \colon \mathbb{H}_n \to \mathbb{B}_n$,

$$\psi(\xi) := \left(-\frac{2\mathrm{i}\xi'}{1 - \mathrm{i}\xi_n}, \frac{1 + \mathrm{i}\xi_n}{1 - \mathrm{i}\xi_n}\right).$$

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Pseudohyperbolyc metric:

$$\rho_{\mathbb{B}_n}(z, w) := |\varphi_z(w)|, \qquad \rho_{\mathbb{H}_n}(\xi, \eta) := \rho_{\mathbb{B}_n}(\psi(\xi), \psi(\eta)).$$

The reproducing kernel over the Siegel domain

$$\begin{tikzcd}
H_1 = \mathcal{A}_m(\mathbb{B}_n, \mu_\alpha) \arrow[rr, "Uf(z) := J(z) f(\psi(z))"] & & H_2 = \mathcal{A}^2_m(\mathbb{H}_n, \nu_\alpha) \\
X = \mathbb{B}_n & & Y = \mathbb{H}_n \arrow[ll, "\psi"']
\end{tikzcd}$$

The reproducing kernel over the Siegel domain

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\begin{tikzcd}
H_1 = \mathcal{A}_m(\mathbb{B}_n, \mu_\alpha) \arrow[rr, "Uf(z) := J(z) f(\psi(z))"] & & H_2 = \mathcal{A}^2_m(\mathbb{H}_n, \nu_\alpha) \\
X = \mathbb{B}_n & & Y = \mathbb{H}_n \arrow[ll, "\psi"']
\end{tikzcd}
$$

$$
K_\xi^{\mathbb{H}_n}(\eta) = \frac{\left(\frac{\xi_n - \overline{\eta_n}}{2\mathrm{i}} - \langle \xi', \eta' \rangle\right)^{m-1}}{\left(\frac{\eta_n - \overline{\xi_n}}{2\mathrm{i}} - \langle \eta', \xi' \rangle\right)^{n+m+\alpha}} R_{m-1}^{(\alpha, n-1)}\left(\rho_{\mathbb{H}_n}(\xi, \eta)^2\right).
$$

Bibliography

Maximenko, E.; Ramos-Vazquez, G.; L-P (2021):
Homogeneously polyanalytic kernels on the unit ball and the Siegel domain.
https://doi.org/10.1007/s11785-021-01145-z

Bibliography

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A simultaneous and independent work by Youssfi:
Polyanalytic reproducing kernels in $\mathbb{C}^n$.
https://hal.archives-ouvertes.fr/hal-03131190

Thank you.