Soliton theory and Hankel operators

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Abstract

Soliton theory and the theory of Hankel (and Toeplitz) operators have stayed essentially hermetic to each other. This talk is concerned with linking together these two very active and extremely large theories. On the prototypical example of the Cauchy problem for the Korteweg-de Vries (KdV) equation we demonstrate the power of the language of Hankel operators in which symbols are conveniently represented in terms of the scattering data for the Schrodinger operator associated with the initial data for the KdV equation. This approach yields short-cuts to already known results as well as to a variety of new ones (e.g. wellposedness beyond standard assumptions on the initial data) which are achieved by employing some subtle results for Hankel operators.

Soliton theory originated in the mid 60s from the fundamental Gardner-Greene-Kruskal-Miura discovery of what we now call the inverse scattering transform (IST) for the KdV equation. In the context of the Cauchy problem for the KdV equation on the full line

$$\partial_t q - 6q\partial_x q + \partial_x^3 q = 0, \tag{1}$$

$$q(x,0) = q(x), \tag{2}$$

the IST method consists, as the standard Fourier transform method, of three steps:

Step 1. (direct transform)

$$q(x) \longrightarrow S_q,$$

where S_q is a new set of variables which turns (1) into a simple first order linear ODE for $S_q(t)$ with the initial condition $S_q(0) = S_q$.

Step 2. (time evolution)

$$S_q \longrightarrow S_q(t).$$

Step 3. (inverse transform)

$$S_q(t) \longrightarrow q(x,t).$$

In the classical IST for (1)-(2), when q is rapidly decaying as $|x| \to \infty$ (the so-called short range), S_q is a set of scattering data associated with the Schrödinger operator

$$\mathbb{L}_q = -\partial_x^2 + q.$$

By solving the Schrödinger equation $\mathbb{L}_q u = k^2 u$ one finds $S_q = \{R(k), (\kappa_n, c_n)\}$, where $R(k)$, $k \in \mathbb{R}$, is the reflection coefficient and (κ_n, c_n), $n = 1, 2, .., N$, are so-called bound state data associated with eigenvalues $-\kappa_n^2$ of $\mathbb{L}_q$. Step 2 readily yields

$$S_q(t) = \left\{ R(k) \exp\left(8ik^3 t\right),\ \kappa_n, c_n \exp\left(8\kappa_n^3 t\right) \right\}. \tag{3}$$

Step 3 amounts to solving the inverse scattering problem of recovering the potential $q\,(x,t)$ (which now depends on $t \geq 0$).

$$q\,(x,t) = -2\partial_x^2 \log\det\left(I + \mathbb{H}\,(x,t)\right), \tag{4}$$

where $\mathbb{H}\,(x,t)$ is the Hankel operator $\mathbb{H}\,(\varphi_{x,t})$ with symbol

$$\varphi_{x,t}(k) = R(k)\xi_{x,t}(k) + \sum_{n=1}^{N} \frac{c_n \xi_{x,t}(i\kappa_n)}{\kappa_n + ik}. \tag{5}$$

Here $\xi_{x,t}(k) = \exp\{i(8k^3 t + 2kx)\}$. Steps 1-3 can now be combined to read

$$q(x) \longrightarrow \mathbb{H}(\varphi_{x,t}) \longrightarrow q(x,t). \tag{6}$$

$$(I + \mathbb{H}(x,t))Y = -\mathbb{H}(x,t)(1)$$

$$q(x,t) = \partial_x \lim 2ikY(k,x), \quad k \to \infty$$

Hypothesis (1)

Let q be a real locally integrable function subject to

(1) *(boundedness from below)*

$$\inf \operatorname{Spec}(\mathbb{L}_q) = -h_0^2 > -\infty; \tag{7}$$

(2) *(decay at $+\infty$) For some positive weight function $w(x) \geq x$*

$$\int^{\infty} w(x)\,|q(x)|\,dx < \infty. \tag{8}$$

We show that under Hypothesis 1 (1)-(2) is globally well-posed and completely integrable in the sense that (6) can be explicitly realized. Note that our class of initial profiles, which we call step-like, is extremely large. The condition

$$\operatorname*{Sup}_{|I|=1} \int_I \max\left(-q\left(x\right), 0\right) \ dx < \infty, \tag{9}$$

is sufficient for (7) and is also necessary if $q \leq 0$. Therefore, any q subject to Hypothesis 1 is essentially bounded from below, decays sufficiently fast at $+\infty$ but is arbitrary otherwise.

The main feature of our situation is that we can do one sided scattering theory and define a suitable (right) reflection coefficient $R(k)$. The problem is that R need not have smoothness and decay properties that the machinery of the classical IST relies on is that Hypothesis 1 does not rule out the case $|R(k)| = 1$ for a.e. real k which further complicates the situation. In the quantum mechanical sense, such q's are completely nontransparent (repulsive) for plane waves coming from $+\infty$.

Hardy space

A function f analytic in $\mathbb{C}^{\pm}$ is in the Hardy space $H^p_{\pm}$ for some $0 < p \leq \infty$ if

$$\|f\|^p_{H^p_{\pm}} \stackrel{\text{def}}{=} \sup_{y>0} \|f(\cdot \pm iy)\|_p < \infty.$$

We remind the reader that by our convention we set $H^p = H^p_+$.

$$\|f\|_{H^p_{\pm}} = \|f\,(\cdot \pm i0)\,\|_{L^p} \stackrel{\text{def}}{=} \|f\|_p\,.$$

Classes $H^\infty_\pm$ and $H^2_\pm$ will be particularly important. $H^\infty_\pm$ is the algebra of uniformly bounded in $\mathbb{C}^\pm$ functions and $H^2_\pm$ is the Hilbert space with the inner product induced from L^2:

$$\langle f, g\rangle_{H^2_\pm} = \langle f, g\rangle_{L^2} = \langle f, g\rangle = \int f\bar{g}.$$

It is well-known that $L^2 = H^2 \oplus H^2_-$, the orthogonal (Riesz) projection $\mathbb{P}_\pm$ onto $H^2_\pm$ being given by

$$(\mathbb{P}_\pm f)(x) = \pm\frac{1}{2\pi i}\lim_{\varepsilon\to 0+}\int \frac{f(s)ds}{s-(x\pm i\varepsilon)} \stackrel{\text{def}}{=} \pm\frac{1}{2\pi i}\int \frac{f(s)ds}{s-(x\pm i0)}. \quad (10)$$

Of course

$$\mathbb{P}^*_\pm = \mathbb{P}_\pm,\ \mathbb{P}^2_\pm = \mathbb{P}_\pm,\ \mathbb{P}_+ + \mathbb{P}_- = \mathbb{I}. \quad (11)$$

Hankel and Toeplitz Operators

Let

$$(\mathbb{J}f)(x) \stackrel{\text{def}}{=} f(-x)$$

be the operator of reflection on L^2. It is clearly an isometry with the obvious properties

$$\mathbb{J}^* = \mathbb{J},\ \mathbb{J}^2 = \mathbb{I},\ \mathbb{J}^{-1} = \mathbb{J}. \tag{12}$$

$$\mathbb{J}\,(\varphi f) = (\mathbb{J}\varphi)\mathbb{J}f,\ \varphi \in L^\infty,\ f \in L^2 \tag{13}$$

$$\mathbb{J}\mathbb{P}_{\mp} = \mathbb{P}_{\pm}\mathbb{J}. \tag{14}$$

Definition (3) (Hankel and Toeplitz operators)

Let $\varphi \in L^\infty$. The operators $\mathbb{H}(\varphi)$ and $\mathbb{T}(\varphi)$ defined by

$$\mathbb{H}(\varphi)f = \mathbb{J}\mathbb{P}_-\varphi f, \text{ and } \mathbb{T}(\varphi)f = \mathbb{P}_+\varphi f, \quad f \in H^2, \tag{15}$$

are called respectively the Hankel and Toeplitz operators with the symbol φ.

Theorem (4) (Widom/Devinatz)

Let φ be unimodular. Then $\|\mathbb{H}(\varphi)\| < 1$ iff $\mathbb{T}(\varphi)$ is left invertible.

Since obviously $\mathbb{H}(\varphi + h) = \mathbb{H}(\varphi)$, for any $h \in H^{\infty}$, only the part of φ analytic in $\mathbb{C}^-$ is essential.

$$\mathbb{H}(\varphi) = \mathbb{H}(\widetilde{\mathbb{P}}_- \varphi)$$

Proposition (5)

$\mathbb{H}(\varphi)$ is selfadjoint if $\mathbb{J}\varphi = \bar{\varphi}$.

In the context of integral operators the Hankel operator is usually defined as an integral operator on $L^2(\mathbb{R}_+)$ whose kernel depends on the sum of the arguments

$$(\mathbb{H}f)(x) = \int_0^\infty h(x+y)f(y)dy, \; f \in L^2(\mathbb{R}_+), \; x \geq 0 \tag{16}$$

and it is this form that Hankel operators typically appear in the inverse scattering formalism. One can show that the Hankel operator $\mathbb{H}$ defined by (16) is unitary equivalent to $\mathbb{H}(\varphi)$ with the symbol φ equal to the Fourier transform of h.

Hankel operators and the Sarason algebra $H^\infty + C$

The set $H^\infty + C$ is one of the most common function classes in the theory of Hankel and Toeplitz operators. By definition

$$H^\infty + C \stackrel{\text{def}}{=} \{f : f = h + g,\ h \in H^\infty,\ g \in C\}.$$

Theorem (6) (Sarason, 1967)

$H^\infty + C$ is a closed sub-algebra of L^∞.

The importance of $H^\infty + C$ in the context of Hankel operators is due to the following fundamental theorem.

Theorem (7) (Hartman, 1958)

Let $\varphi \in L^\infty$. Then $\mathbb{H}(\varphi)$ is compact iff $\varphi \in H^\infty + C$. I.e. $\mathbb{H}(\varphi)$ is compact iff $\mathbb{H}(\varphi) = \mathbb{H}(g)$ with some $g \in C$.

Theorem (8) (Grudsky, 2001)

Let $p(x)$ be a real polynomial with a positive leading coefficient such that

$$p(-x) = -p(x). \tag{17}$$

Then

$$e^{ip} \in H^\infty + C. \tag{18}$$

Moreover, there exist an infinite Blaschke product B and a unimodular function $u \in C$ such that

$$e^{ip} = Bu. \tag{19}$$

It is worth mentioning that this theorem is a particular case of a more general statement. This statement says that Theorem 7 holds not only for polynomial but any function f such that $f(-x) = -f(x)$ and

$$\lim_{x\to\infty} \inf \frac{xf''(x)}{f'(x)} > -2, \lim_{x\to\infty} \frac{xf''(x)}{f'(x)^2} = 0, \lim_{x\to\infty} \frac{\sqrt{x}f''(x)}{f'(x)^{3/2}} = 0.$$

We emphasize that functions of the form e^{ip} commonly appear in the IST approach to completely integrable PDEs. For instance, in the KdV case

$$p(\lambda) = t\lambda^3 + x\lambda$$

with real x (spatial variable) and positive t (time).

Definition (9)

A function $f \in H^\infty + C$ is said to be invertible in $H^\infty + C$ if $1/f \in H^\infty + C$. Similarly, f is not invertible in $H^\infty + C$ if $1/f \notin H^\infty + C$.

Theorem (10)

Let $\varphi \in H^\infty + C$ and $1/\varphi \in L^\infty$. Then

$$1/\varphi \notin H^\infty + C \Longrightarrow \mathbb{T}(\varphi) \text{ is left-invertible,} \tag{20}$$

$$1/\varphi \in H^\infty + C \Longrightarrow \mathbb{T}(\varphi) \text{ is Fredholm.} \tag{21}$$

Theorem (11)

If $\varphi \in H^\infty + C$ *and unimodular but not invertible in* $H^\infty + C$*, then*

$$\|\mathbb{H}(\varphi)\| < 1. \tag{22}$$

Theorem (12)

If $u \in H^\infty + C$*,* $|u| = 1$*, and* p *is as in Theorem 7, then*

$$\|\mathbb{H}(e^{ip}u)\| < 1.$$

Theorem (13)

If $\varphi \in H^\infty + C$ *is not unimodular but* $\|\varphi\|_\infty \leq 1$ *and* $\mathbb{J}\varphi = \bar{\varphi}$ *then (22) holds.*

The classical IST and Hankel operators

Through this section we assume that the initial profile q in (1)-(2) is real and short range, i.e. $(1+|x|)\,q\,(x) \in L^1$. In the sequel we refer to such initial data as classical.

Direct scattering problem. Associate with q the full line Schrödinger operator $\mathbb{L}_q = -\partial_x^2 + q(x)$. As is well-known, $\mathbb{L}_q$ is self-adjoint on L^2 and

$$\mathrm{Spec}(\mathbb{L}_q) = \{-\kappa_n^2\}_{n=1}^N \cup \mathbb{R}_+.$$

The singular spectrum of $\mathbb{L}_q$ consists of a finite number of simple negative eigenvalues $\{-\kappa_n^2\}$, called bound states, and absolutely continuous (a.c.) two fold component filling $\mathbb{R}_+$. There is no singular continuous spectrum. Two linearly independent (generalized) eigenfunctions of the a.c. spectrum $\psi_\pm(x,k)$, $k \in \mathbb{R}$, can be chosen to satisfy

$$\psi_\pm(x,k) = e^{\pm ikx} + o(1),\ \partial_x\psi_\pm(x,k) \mp ik\psi_\pm(x,k) = o(1), \quad x \to \pm\infty. \tag{23}$$

The functions $\psi_\pm$ are referred to as Jost solutions of the Schrödinger equation

$$\mathbb{L}_q\psi = k^2\psi. \tag{24}$$

Theorem (14) (On Jost solutions)

The Jost solutions $\psi_\pm(x,k)$ *are analytic for* $\operatorname{Im} k > 0$ *and continuous for* $\operatorname{Im} k \geq 0$. *Moreover as* $k \to \infty$, $\operatorname{Im} k \geq 0$,

$$\psi_\pm(x,k) = e^{\pm ikx}\left(1 \pm \frac{i}{2k}\int_x^{\pm\infty} q + O\left(\frac{1}{k^2}\right)\right) \tag{25}$$

and

$$\psi_\pm(x,-k) = \overline{\psi_\pm(x,k)}, \;\; k \in \mathbb{R}. \tag{26}$$

To remove the oscillatory behavior of $\psi_\pm$ let us introduce the functions, sometimes called Faddeev functions,

$$y_\pm(k, x) := e^{\mp ikx}\psi_\pm(x, k). \tag{27}$$

The function $y := y_+$ will be used more frequently.

Corollary (15)

For some a large enough

$$y(\cdot, a)^{\pm 1} \in H^\infty \cap C. \tag{28}$$

Since q is real, $\overline{\psi_\pm}$ also solves (24) and one can easily see that the pairs $\{\psi_+, \overline{\psi_+}\}$ and $\{\psi_-, \overline{\psi_-}\}$ form fundamental sets for (24). Hence $\psi_\mp$ is a linear combination of $\{\psi_\pm, \overline{\psi_\pm}\}$. Elementary Wronskian considerations then yield the (basic) right/left scattering relations

$$T(k)\psi_\mp(x,k) = \overline{\psi_\pm(x,k)} + R_\pm(k)\psi_\pm(x,k), \;\; k \in \mathbb{R}, \tag{29}$$

where $T, R_\pm$, called the transmission and right/left reflection coefficients respectively. It immediately follows from (29) that ($\operatorname{Im} k = 0$)

$$T = \frac{2ik}{W(\psi_-, \psi_+)}, \tag{30}$$

$$R_+ = \frac{W(\overline{\psi_+}, \psi_-)}{W(\psi_-, \psi_+)}, \quad R_- = \frac{W(\psi_+, \overline{\psi_-})}{W(\psi_-, \psi_+)}, \tag{31}$$

where the Wronskians ($W(f,g) = fg' - f'g$) are independent of x (∂_x is missing in $\mathbb{L}_q$).

Theorem (16)

The transmission coefficient $T \in C$ *and is analytic in* $\mathbb{C}^+$ *except for a finite number of simple poles* $\{i\kappa_n\}_{n=1}^N$ *with the residues*

$$\operatorname{Res}(T, i\kappa_n) = i\mu_n^{\pm 1} c_n^{\pm}, \tag{32}$$

where $c_n^{\pm}$ *are norming constants defined by (34) and* μ_n *determined from*

$$\psi_+(x, i\kappa_n) = \mu_n \psi_-(x, i\kappa_n). \tag{33}$$

Moreover,

$$\lim T(k) = 1,\ k \to \infty,\ \operatorname{Im} k \geq 0.$$

$$c_n^{\pm} \stackrel{\text{def}}{=} \|\psi_{\pm}(\cdot, i\kappa_n)\|_2^{-2} \tag{34}$$

The reflection coefficients $R_{\pm} \in C$ *(but need not be analytic),* $|R(k)| < 1$ *for* $k \neq 0$ *and generically*$R(0) = -1$*. Furthermore,*

$$T(-k) = \overline{T(k)}, \quad R_{\pm}(-k) = \overline{R_{\pm}(k)}, \quad |T(k)|^2 + |R(k)|^2 = 1, \quad k \in \mathbb{R}. \tag{35}$$

$$Ty_- = \bar{y}_+ + R\xi_x y_+ \tag{36}$$

Let us regard (36) as a Hilbert-Riemann problem of determining $y_\pm$ by given T, R which we will solve by Hankel operator techniques. The potential q can then be easily found by (25).

$$\xi_(x) = e^{i2kx}$$

Note now that for each fixed x

$$T\,(k)\,y_-\,(k,x) - 1 - \sum_{n=1}^{N} \frac{ic_n^+ \xi_x(i\kappa_n)}{k - i\kappa_n} y(i\kappa_n, x) \in H^2.$$

Abbreviating $R_x := R\xi_x$, $c_{x,n} := c_n^+ \xi_x(i\kappa_n)$, rewrite (36) in the form

$$
\begin{aligned}
& T(k)\, y_-(k,x) - 1 - \sum_{n=1}^{N} \frac{ic_{x,n}}{k - i\kappa_n} y(i\kappa_n, x) \\
& = \overline{(y(k,x) - 1)} + R_x(k)\,(y(k,x) - 1) \\
& + R_x(k) - \sum_{n=1}^{N} \frac{ic_{x,n}}{k - i\kappa_n} y(i\kappa_n, x) .
\end{aligned} \tag{37}
$$

Applying $\mathbb{P}_-$ to both sides of (37) and defining $Y := (y(k,x) - 1)$ we get

$$\mathbb{J}Y + \mathbb{P}_- \left(R_x - \sum_{n=1}^{N} \frac{ic_{x,n}}{\cdot - i\kappa_n} \right) Y = -\mathbb{P}_- \left(R_x - \sum_{n=1}^{N} \frac{ic_{x,n}}{\cdot - i\kappa_n} \right).$$

Applying $\mathbb{J}$ to both sides of this equation yields

$$(\mathbb{I} + \mathbb{H}(\varphi))Y = -\mathbb{H}(\varphi)1, \tag{38}$$

where $\mathbb{H}(\varphi)$ is the Hankel operator defined in Definition 2 with symbol

$$\varphi(k) = \varphi_x(k) = R(k)\xi_x(k) + \sum_{n=1}^{N} \frac{c_n \xi_x(i\kappa_n)}{\kappa_n + ik}$$

where x is a real parameter $\left(\xi_x(k) = e^{2ikx}\right)$.

Inverse scattering transform. To reformulate the classical IST in terms of Hankel operators, we recall the classical fact that the initial short range profile q in (1)-(2) evolves under the KdV flow in such a way that the scattering data $S_q(t)$ for $q(x,t)$ evolves by (3). It is convenient to introduce

$$S_q(x,t) \stackrel{\text{def}}{=} \left\{ R(k)\xi_{x,t}(k),\ k \geq 0,\ (\kappa_n, c_n \xi_{x,t}(i\kappa_n))_{n=1}^{N} \right\},$$

the time evolved scattering data corresponding to the shifted initial profile $q(\cdot + x)$.

$$\xi_{x,t}(k) = e^{i(8tk^3 + 2xk)}$$

Observe that the KdV flow preserves at least the Schwartz class. It is remarkable that if one solves (38) with

$$\varphi = \varphi_{x,t}(k) = R(k)\xi_{x,t}(k) + \sum_{n=1}^{N} \frac{c_n \xi_{x,t}(i\kappa_n)}{\kappa_n + ik} \tag{39}$$

by the Fredholm series formula then $q(x,t)$ computed by (41) simplifies to

$$q(x,t) = -2\partial_x^2 \log\det(\mathbb{I} + \mathbb{H}(\varphi_{x,t})), \tag{40}$$

$$q(x,t) = \partial_x \lim 2ikY(k,x), \quad k \to \infty. \tag{41}$$

The general step-like case. A real-valued locally integrable potential q is said to be Weyl limit point at $\pm\infty$ if the equation $\mathbb{L}_q u = \lambda u$ has a unique (up to a multiplicative constant) solution such that $\Psi_\pm(\cdot, \lambda) \in L^2(a, \pm\infty)$ for each $\lambda \in \mathbb{C}^+$. Such $\Psi_\pm$ is commonly called the Weyl solution on $(a, \pm\infty)$. If $q \in L^1$ then the Weyl solutions $\Psi_\pm(x, \lambda) \sim e^{\pm i\sqrt{\lambda}x}$, $x \to \pm\infty$, clearly turn into Jost and we have

$$\Psi_\pm\left(x, k^2\right) = \psi_\pm\left(x, k\right).$$

However Weyl solutions exist under much more general conditions on q's and no decay of any kind is required. There is no criterion for the limit point case in terms of q (a major unsolved problem) but there are a number of sufficient conditions which are typically satisfied in most of realistic situations. For instance, any q subject to Hypothesis 1 is in the limit point case at $\pm\infty$.

Let us introduce now the right reflection coefficient for potentials subject to Hypothesis 1. Since $W\left(\psi_+, \overline{\psi_+}\right) = -2ik$ the pair $\{\psi_+, \overline{\psi_+}\}$ forms a fundamental set for $\mathbb{L}_q u = k^2 u$ and hence the Weyl solution Ψ_- is a linear combination of $\{\psi_+, \overline{\psi_+}\}$. I.e. for any real $k \neq 0$

$$T(k)\Psi_-(x, k^2) = \overline{\psi_+(x, k)} + R(k)\psi_+(x, k), \tag{42}$$

holds with some T and R. In analogy with (29) we call (42) the (right) basic scattering relation and similarly to (30) we introduce

Definition (20) (Reflection coefficient)

We call

$$R(k) = \frac{W(\overline{\psi_+}(\cdot, k), \Psi_-(\cdot, k^2))}{W(\Psi_-(\cdot, k^2), \psi_+(\cdot, k))} \qquad (43)$$

the (right) reflection coefficient.

Proposition (21) (Properties of the reflection coefficient)

The reflection coefficient R is symmetric $R(-k) = \overline{R(k)}$ and contractive $|R(k)| \leq 1$ a.e. Moreover, if $\sigma(\mathbb{L}_q)$ is the minimal support of the two fold a.c. spectrum of $\mathbb{L}_q$ then $|R(k)| < 1$ for a.e. real k such that $k^2 \in \sigma(\mathbb{L}_q)$ and $|R(k)| = 1$ otherwise.

The IST Hankel Operator

The previous section suggests that the Hankel operator arising in the IST has a very specific structure. In this section we state and prove some of its properties of principal importance.

Definition (25) (IST Hankel operator)

Assume that initial data q is subject to Hypothesis 1. Let R is given by (43) and ρ is positive measure such that

$$\int_0^{h_0} d\rho < \infty.$$

We call the Hankel operator

$$\mathbb{H}(x,t) := \mathbb{H}(\varphi_{x,t}),$$

with the symbol

$$\varphi_{x,t}(k) = \xi_{x,t}(k)R(k) + \int_0^{h_0} \frac{\xi_{x,t}(is)\, d\rho(s)}{s+ik}, \tag{44}$$

the IST Hankel operator associated with q.

Let $q_n => g$ in sense of uniform convergence on compact sets. Then

$$H\left(\varphi^{(n)}x, t\right) \to H\left(\varphi_{x,t}\right) : \text{strong convergence}$$

$$g_n(x) := \mathcal{X}_{(-\infty,n)}(x) \to \text{satisfies classical conditions.}$$

Theorem (26) (Fundamental properties of the IST Hankel operator)

Under Hypothesis 1 the IST Hankel operator $\mathbb{H}(x,t)$ *is well-defined and has the properties: for any* $x \in \mathbb{R},\ t > 0$

(1) $\mathbb{H}(x,t)$ *is selfadjoint,*

(2) $\mathbb{H}(x,t)$ *is compact,*

(3) $\mathbb{I} + \mathbb{H}(x,t) > 0.$

Symbol of Hankel Operator

$$\varphi_x(\lambda) = T(\lambda)\; G_-(\lambda)\; e^{i\Phi(\lambda,x)}. \tag{45}$$

Here

$$\Phi(\lambda, x) = 8t\lambda^3 + 2x\lambda,\; t > 0,\; x \in \mathbb{R}. \tag{46}$$

The function $G_-(\lambda)$ can be represented as the Fourier integral over the half-axis:

$$G_-(\lambda) = \int\limits_0^\infty e^{-2i\lambda s} g(s)\, ds, \tag{47}$$

where $g(s) \in L_1\left(\mathbb{R}_+, (1+s)^\alpha\right)$, is nonegative-valued almost everywhere, i.e.

$$\int\limits_0^\infty g(s)(1+s)^\alpha ds < \infty, \quad \alpha \geq 0. \tag{48}$$

$$T(\lambda) \in H^\infty(\Pi).$$

Main Result

Let $\mathfrak{S}_1$ denote the set of all trace-class operators acting on the space $H^2(\Pi)$. Recall that a compact operator A belong to $\mathfrak{S}_1$, if the sequence of its singular numbers $\{s_j(A)\}_{j=1}^{\infty}$ is summable. The norm of an operator A in $\mathfrak{S}_1$ is defined as

$$\|A\|_{\mathfrak{S}_1} := \sum_{j=1}^{\infty} |s_j(A)|.$$

Along with the Hankel operator we consider its derivatives with respect to the parameter x. It is easy to see that

$$\frac{\partial^j}{\partial x^j}\mathbb{H}(\varphi_x) = \mathbb{H}(\varphi_{j,x}), \tag{49}$$

where

$$\varphi_{j,x}(\lambda) = (2i)^j \lambda^j \varphi_x(\lambda), \quad j = 0, 1, 2, \ldots. \tag{50}$$

Theorem

If the function $\varphi_x(\lambda)$ is of the form (45)-(48) with $g(s) \in L_1\left(\mathbb{R}_+(1+s)^{j/2}\right)$, $j \in \mathbb{N}$, then

$$\frac{\partial^k}{\partial x^k}\mathbb{H}(\varphi_x) \in \mathfrak{S}_1, \quad k = 0, 1, \dots j,$$

and

$$\left\| \frac{\partial^k}{\partial x^k}\mathbb{H}(\varphi_x) \right\|_{\mathfrak{S}_1} \leq \begin{cases} L_1, & x > 0 \\ L_2\,(1+|x|)^{k/2}, & x < 0, \end{cases}$$

where the constants L_1 and L_2 are independent of $x \in \mathbb{R}$.

Peller‘s Theorem

We say that a function $f(\xi)$ analytic in Π belongs to the space $A_1^1(\Pi)$ if and only if

$$\|f\|_{A_1^1(\Pi)} := \int_0^\infty \int_{-\infty}^\infty |f''(\xi_1 + i\xi_2)|\, d\xi_1\, d\xi_2 + \sup\{f(\xi)|\xi_2 \geq 1\} < \infty,$$

where $\xi = \xi_1 + i\xi_2$ is a complex variable belonging to the complex plane $\mathbb{C}$. we introduce the following modification of an analytic projection:

$$\left(\widetilde{P^+} f\right)(\xi) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{1}{\tau - \xi} - \frac{\tau}{1+\tau^2} \right) f(\tau)\, d\tau.$$

Theorem (V.Peller, 1980)

Let $\varphi \in L_\infty(\mathbb{R})$, *Then* $\mathbb{H}(\varphi) \in \mathfrak{S}_1$ *if and only if*

$$\left(\widetilde{P^+\overline{\varphi}}\right)(\xi) \in A_1^1(\Pi).$$

Applying Pellier's Theorem to the Hankel operator with this symbol of the form, we must first estimate the integrals

$$I_j(\xi, x) := \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{1}{\tau-\xi} - \frac{\tau}{1+\tau^2} \right) \tau^j \overline{G_-(\tau)} e^{-i\Phi(\tau,x)}\, d\tau,$$
$$\xi \in \Pi,\ j = 0, 1, 2, \ldots,$$

$$I_j^{(2)}(\xi, x) := \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\tau^j \overline{G_-(\tau)} e^{-i\Phi(\tau,x)}}{(\tau-\xi)^3}\, d\tau, \quad \xi \in \Pi,\ j = 0, 1, 2, \ldots. \quad (51)$$

Changing the order of integration, we obtain

$$I_j(\xi, x) = \frac{1}{2} \int_0^\infty g(s) J_j(s, \xi, x)\, ds,$$

where

$$J_j(s, \xi, x) := \frac{1}{\pi i} \int_{-\infty}^{\infty} \left(\frac{1}{\tau - \xi} - \frac{\tau}{1 + \tau^2} \right) \tau^j e^{-i\Phi(\tau, x-s)}\, d\tau,$$

$$\Phi(\tau, x - s) = 8t\tau^3 + 2(x - s)\tau.$$

Let us make the following change of variables

$$\tau = \beta(s)u, \quad \xi = \beta(s)\xi', \quad \text{where} \quad \beta(s) = \left(\frac{(s-x)}{12t}\right)^{1/2}.$$

Setting

$$S(u) = \frac{u^3}{3} - u, \quad \Lambda(s,x) := \Lambda(s) := \frac{(s-x)^{3/2}}{(3t)^{1/2}}$$

we obtain

$$J_j(s,\xi,x) := \widetilde{J}_j(s,\xi',x) = \beta^j(s)\widetilde{I}_j(s,\xi',x) - \beta^{j+2}(s)\widehat{I}_j(s,x),$$

where

$$\widetilde{I}_j(s,\xi',x) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{u^j e^{-i\Lambda(s)S(u)}}{u-\xi'}\,du \tag{52}$$

$$\widehat{I}(s,x) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{u^{j+1} e^{-i\Lambda(s)S(u)}}{1+\beta^2(s)u^2}\,du. \tag{53}$$

Global Classical Solution of KDV

1. $\dfrac{\partial^{n+m}}{\partial x^n \partial t^m} \mathbb{H}\left(\Phi_{x,t}\right) \in \mathfrak{S}_1.$
2. Main Theorem implies:
   For the operator $\mathbb{H}(\xi_{x,t} R_0)$, we proved that if

$$\int^{\infty} (1 + |s|)^N \, |q(s)| \, ds < \infty,$$

then

$$\frac{\partial^{n+m}}{\partial x^n \partial t^m} \mathbb{H}\left(\xi_{x,t} R_0\right) \in \mathfrak{S}_1$$

for all n and m, satisfying the condition

$$n + 3m \leq 2N - 1.$$

Theorem

Suppose that the (real) initial profile q satisfies the condition

$$\inf \operatorname{Spec}(\mathbb{L}_q) = -a^2 > -\infty \qquad \textit{(is bounded below)},$$

$$\int^{\infty} (1+|x|)^N \, |q(x)| \, dx < \infty, \quad N \geq 1 \quad (\textit{decreases} + \infty).$$

Then the function $\tau(x,t) := \det(1 + \mathbb{H}(x,t))$ *is well defined on* $\mathbb{R} \times \mathbb{R}_+$, *and its classical derivatives* $\partial^{n+m}\tau(x,t) \,/\, \partial x^n \partial t^m$ *exist provided that* $n + 3m \leq 2N - 1$. *Moreover, for* $N \geq 3$ *the Cauchy problem has a global (in time) classical solution which is given by*

$$u(x,t) = -2\frac{\partial^2}{\partial x^2} \log \tau(x,t), \quad t > 0.$$