Toeplitz Operators with symmetric symbols in the Bergman space of the unit ball

International Workshop on Operator Theory on Function Spaces

Armando Sánchez Nungaray.

Universidad Veracruzana
Joint work with Carlos González and Jose Rosales.

September 2022, IWOTFS

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Main classification result:

Grudski,Vasilevski,Quiroga-Barranco

A C^*-algebra generated by the Toeplitz operators is commutative on weighted Bergman space if and only if there is a pencil of hyperbolic geodesics such that the symbols of the Toeplitz operators are constant on the cycles of this pencils.
All cycles are in fact the orbits of the one-parameter subgroup of isometries for the hyperbolic geometry on the unit disk.

This provides us with the following scheme: the C^*-algebra generated by the Toeplitz operators is commutative on each weighted Bergman space if and only if there is a maximal commutative subgroup of the Möbius transformation such that the symbols of the Toeplitz operators are invariant under the action of this subgroup.

Vasilevski,Quiroga-Barranco

For the unit ball $\mathbb{B}^n$ there exist $n+2$ different cases of commutative C^*-algebras generated by Toeplitz operators, acting on weighted Bergman spaces. In all cases the bounded measurable symbols of Toeplitz operators are invariant under the action of maximal abelian commutative subgroups of biholomorphisms of the unit ball.

Maximal Abelian subgroups

Quasi-elliptic action: The $\mathbb{T}^n$-action on $\mathbb{B}^n$ given by

$$t \cdot z = (t_1 z_1, \ldots, t_n z_n).$$

Quasi-parabolic action: The $\mathbb{T}^{n-1} \times \mathbb{R}$-action on D_n given by

$$(t', x) \cdot z = (t' z', z_n + x).$$

Quasi-hyperbolic action: The $\mathbb{T}^{n-1} \times \mathbb{R}_+$-action on D_n given by

$$(t', r) \cdot z = (r^{\frac{1}{2}} t' z', r z_n).$$

Nilpotent action: The $\mathbb{R}^n$-action on D_n given by

$$b \cdot z = (z' + b', z_n + b_n + 2i z' \cdot b' + i|b'|^2).$$

Quasi-nilpotent action: For every integer $k = 1, \ldots, n-2$, the $\mathbb{T}^k \times \mathbb{R}^{n-k}$-action on D_n given by

$$(t, b) \cdot z = (t z_{(1)}, z_{(2)} + b', z_n + b_{n-k} + 2i z_{(2)} \cdot b' + i|b'|^2).$$

Theorem (Vasilevski and Quiroga-Barranco)

Let a be a bounded measurable MASG function. Then Toeplitz operator T_a acting on $\mathcal{A}^2_\lambda(D)$ is unitary equivalent to multiplication operators γ_a,

$$R\, T_a R^* = \gamma_a I,$$

acting on H.

Theorem (Vasilevski and Quiroga-Barranco)

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Quasi-Elliptic

$$\gamma_a(p) = \frac{(2\pi)^n \Gamma(n+|p|+\lambda+1) c_{\lambda,n}}{2^n p! \Gamma(n+\lambda+1)} \int_{\tau(\mathbb{B}^n)} a\left(\sqrt{r}\right) r^{2p} (1-r^2)^\lambda r \mathrm{d}\, r,$$

where $\tau(\mathbb{B}^n) = \{r \in \mathbb{R}^n_+ : r_1^2 + \cdots r_n^2 < 1\}$.

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Let G be MASG of biholomorphisms of either $\mathbb{B}^n$ or D_n, so that its Lie algebra can be identified with $\mathbb{R}^n$. For every $X \in \mathbb{R}^n$, the G-action induces a smooth vector field given by

$$X_z^\sharp = \frac{\mathrm{d}}{\mathrm{d}\, t}\,|_{s=0} \exp(sX) \cdot z$$

for every z in the corresponding domain, where $\cdot$ denotes the G-action on such domain.

Moment Map

Let us denote by D either of the domains $\mathbb{B}^n$ or D_n, and let G be a MASG of biholomorphisms of D. A moment map for the G-action is a smooth G-invariant map $\mu = \mu^G : D \to \mathbb{R}^n$ such that

$$\mathrm{d}\, \mu_X = \omega(X^\sharp, \cdot)$$

for every $X \in \mathbb{R}^n$, where ω is the Kähler form of D and $\mu_X : D \to \mathbb{R}$ is the smooth function given by

$$\mu_X(z) = \langle \mu(z), X \rangle,$$

Symplectic Form

Unit Ball

$$(\omega_{\mathbb{B}^n})_z = i \sum_{j,k=1}^{n} \frac{(1-|z|^2)\delta_{jk} + \overline{z}_j z_k}{(1-|z|^2)^2} \mathrm{d}\, z_j \wedge \mathrm{d}\, \overline{z}_k$$
$$= \frac{i}{(1-|z|^2)^2} \left((1-|z|^2) \sum_{j=1}^{n} \mathrm{d}\, z_j \wedge \mathrm{d}\, \overline{z}_j + \sum_{j,k=1}^{n} \overline{z}_j z_k \mathrm{d}\, z_j \wedge \mathrm{d}\, \overline{z}_k \right),$$

Siegel Domain

$$(\omega_{D_n})_z$$
$$= \frac{i}{(\mathrm{Im}(z_n) - |z'|^2)^2} \left((\mathrm{Im}(z_n) - |z'|^2) \sum_{j=1}^{n-1} \mathrm{d}\, z_j \wedge \mathrm{d}\, \overline{z}_j + \sum_{j,k=1}^{n-1} \overline{z}_j z_k \mathrm{d}\, z_j \wedge \mathrm{d}\, \overline{z}_k \right.$$
$$\left. + \frac{1}{2i} \sum_{j=1}^{n-1} (\overline{z}_j \mathrm{d}\, z_j \wedge \mathrm{d}\, \overline{z}_n - z_j \mathrm{d}\, z_n \wedge \mathrm{d}\, \overline{z}_j) + \frac{1}{4} \mathrm{d}\, z_n \wedge \mathrm{d}\, \overline{z}_n \right),$$

Moment Map associated to MASG

Quasi-elliptic moment map: For the $\mathbb{T}^n$-action on $\mathbb{B}^n$

$$\mu(z) = -\frac{1}{1-|z|^2}(|z_1|^2, \dots, |z_n|^2).$$

Quasi-parabolic moment map: For the $\mathbb{T}^{n-1} \times \mathbb{R}$-action on D_n

$$\mu(z) = -\frac{1}{2(\mathrm{Im}(z_n) - |z'|^2)}(2|z_1|^2, \dots, 2|z_{n-1}|^2, 1).$$

Quasi-hyperbolic moment map: For the $\mathbb{T}^{n-1} \times \mathbb{R}_+$-action on D_n

$$\mu(z) = -\frac{1}{2(\mathrm{Im}(z_n) - |z'|^2)}(2|z_1|^2, \dots, 2|z_{n-1}|^2, \mathrm{Re}(z_n)).$$

Nilpotent moment map: For the $\mathbb{R}^n$-action on D_n

$$\mu(z) = -\frac{1}{2(\mathrm{Im}(z_n) - |z'|^2)}(-4\mathrm{Im}(z'), 1).$$

Quasi-nilpotent moment map: For the $\mathbb{T}^k \times \mathbb{R}^{n-k}$-action on D_n, where $k = 1, \ldots, n-2$

$$\mu(z) = -\frac{1}{2(\mathrm{Im}(z_n) - |z'|^2)}(2|z_1|^2, \ldots, 2|z_k|^2, -4\mathrm{Im}(z_{(2)}), 1).$$

Theorem (Quiroga-Barranco and SN)

Let D denote either of the domains $\mathbb{B}^n$ or D_n and let H be a MASG of the group of biholomorphisms of D. Suppose that $\mu : D \to \mathbb{R}^n$ denotes the moment map for the H-action. Then, the C^*-algebra $\mathcal{T}^{(\lambda)}(L^\infty(D)^\mu)$ generated by Toeplitz operators acting on $\mathcal{A}^2_\lambda(D)$ whose symbols are μ-functions (i.e. that belong to $L^\infty(D)^\mu$) is commutative for every $\lambda > -1$.

Proposition

Let G be a MASG of biholomorphisms of D from the list in Moment Maps and let H be a connected Abelian subgroup of G. Then, the H-action on D has a moment map given by

$$\mu^H : D \to \mathfrak{a}$$
$$\mu^H = \iota^* \circ \mu^G,$$

where $\mu^G : D \to \mathbb{R}^n$ is the moment map for the G-action on D and $\iota^* : \mathbb{R}^n \to \mathfrak{a}$ is the orthogonal projection.

Theorem (Quiroga-Barranco and SN)

Let D be either of the domains $\mathbb{B}^n$ or D_n and let H be a connected Abelian subgroup of the group of biholomorphisms of D with moment map μ. Then, for every $\lambda > -1$, the C^*-algebra $\mathcal{T}^{(\lambda)}(L^\infty(D)^\mu)$ generated by Toeplitz operators whose symbols are essentially bounded μ-functions is commutative. Furthermore, if G is a MASG containing H, then we have the inclusion

$$\mathcal{T}^{(\lambda)}(L^\infty(D)^\mu) \subset \mathcal{T}^{(\lambda)}(L^\infty(D)^G)$$

for every $\lambda > -1$, where $\mathcal{T}^{(\lambda)}(L^\infty(D)^G)$ is the C^*-algebra generated by Toeplitz operators with G-invariant symbols.

Theorem (Quiroga-Barranco and SN)

Let D denote either of the domains $\mathbb{B}^n$ or D_n. Then, for every $\lambda > -1$ the assignment

$$H \mapsto \mathcal{T}^{(\lambda)}(L^{\infty}(D)^{\mu^H})$$

is an inclusion preserving map from the set of connected Abelian subgroups of the group of biholomorphisms of D into the set of commutative C^*-algebras of operators acting on $\mathcal{A}^2_{\lambda}(D)$.

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General Result

Theorem (Olafsson, Dawson, Quiroga-Barranco)

Let H be a closed subgroup of G and let us denote by $\mathcal{A}^H$ the subspace of $L^\infty(D)$ that consists of the H-invariant bounded symbols on D. If for some $\lambda > p-1$ the algebra $\mathrm{End}_H(\mathcal{H}^2_\lambda(D))$ is commutative, then $\mathcal{T}^{(\lambda)}(\mathcal{A}^H)$ is a commutative C^*-algebra. In particular, the result holds if H is a type I group, in the sense of von Neumann algebras, and the restriction $\pi_\lambda\mid_H$ is multiplicity-free.

Notation and Definitions

Let H be a Lie group and π a unitary representation on a Hilbert space $\mathcal{H}$. Suposse that $\mathcal{H}$ contains a dense subspace V that can be algebraically decomposed as

$$V = \sum_{i \in J} \mathcal{H}_j,$$

where the subspaces $\mathcal{H}_j$ are mutually orthogonal, close in $\mathcal{H}$ and irreducible H.invariant modules. Then if H_{j_1} and H_{j_2} are not isomorphic as H-modules for $j_1 \neq j_2$, then the algebra of intertwining operators, denoted by $\mathrm{End}_H(\mathcal{H})$, is conmutative.

Choose $(e_l)_{l \in L}$ an orthonormal basis of $\mathcal{H}$ for which we have a disjoint union

$$L = \bigcup_{i \in J} L_j,$$

so that for every $j \in J$ the set $(e_l)_{l \in L_j}$ is an orthonormal base for $\mathcal{H}_j$.

Toeplitz H invariant unit ball

Theorem

Let H be closed subgroup of $U(n)$. If $a \in L^{\infty}(\mathbb{B}^n)$ is H invariant, in other words, if it satisfies $a \circ A = a$ for every $A \in H$, then, for every $\alpha > -1$, we have $T_a \in End_H(\mathcal{H}^2_\alpha(\mathbb{B}^n))$.

$\mathbb{T}^n$-intertwining operators

Proposition (Quiroga-Barranco)

The decomposition of $\mathcal{P}(\mathbb{C}^n)$ into irreducible $\mathbb{T}^n$-modules is given by

$$\mathcal{P}(\mathbb{C}^n) = \bigoplus_{m \in \mathbb{N}^n} \mathbb{C} z^m$$

More precisely, for every $m \in \mathbb{N}^n$, the space $\mathbb{C}z^m$ is an irreducible $\mathbb{T}^n$-submodule. Moreover, for $m, m' \in \mathbb{N}^n$ we have $\mathbb{C}z^m \not\cong \mathbb{C}z^{m'}$ as $\mathbb{T}^n$-modules and $\mathbb{C}z^m \perp \mathbb{C}z^{m'}$ whenever $m \neq m'$.
In particular, for every $\alpha > -1$ we have

$$H^2_\alpha(\mathbb{B}^n) = \bigoplus_{m \in \mathbb{N}^n} \mathbb{C} z^m$$

as an orthogonal direct sum of Hilbert spaces that yields the decomposition of $H^2_\alpha(\mathbb{B}^n)$ into irreducible $\mathbb{T}^n$-modules.

$\mathbb{T}^n$-intertwining operators

Theorem (Quiroga-Barranco)

For every $\alpha > -1$, the algebra $End_{\mathbb{T}^n}(H^2_\alpha(\mathbb{B}^n))$ is commutative. More precisely, with the above notation and for the unitary map

$$R : H^2_\alpha(\mathbb{B}^n) \longrightarrow l^2(\mathbb{Z}^n_+)$$
$$R(f) = (\langle f, e_m \rangle)_{m \in \mathbb{Z}^n_+},$$

every operator $T \in End_{\mathbb{T}^n}(H^2_\alpha(\mathbb{B}^n))$ is unitarily equivalent to RTR^* which is the multiplication operator on $l^2(\mathbb{Z}^n_+)$ by the function

$$\gamma_T : \mathbb{Z}^n_+ \longrightarrow \mathbb{C}$$
$$\gamma_T(m) = \langle Te_m, e_m \rangle$$

Toeplitz operator with separately radial symbols

Theorem

The C^*-algebra generated by Toeplitz operators with separately radial symbols is commutative, i. e. , for every $\alpha > -1$ the Toeplitz operator is unitary equivalent to

$$RTR^* = \gamma_{a,\alpha}(m)I$$

action on $l_2(\mathbb{Z}_+^n)$, and the spectral function is given by

$$\gamma_{a,\alpha}(m) = \frac{2^n\Gamma(n+|m|+\alpha+1)}{m!\Gamma(n+\alpha+1)}\int_{\tau(\mathbb{B}^n)} a(r)r^{2m}(1-|r|^2)^{\alpha}\prod_{k=1}^{n} r_k d_k.$$

$U(n)$-intertwining operators

Proposition (Quiroga-Barranco)

Let us denote by $\mathcal{P}^k(\mathbb{C}^n)$ the space of homogeneous polynomials on $\mathbb{C}^n$ of degree k. The decomposition of $\mathcal{P}(\mathbb{C}^n)$ into irreducible $U(n)$-modules is given by

$$\mathcal{P}(\mathbb{C}^n) = \bigoplus_{k\in\mathbb{N}} \mathcal{P}^k(\mathbb{C}^n)$$

More precisely, for every $k \in \mathbb{N}$, the space $\mathcal{P}^k(\mathbb{C}^n)$ is an irreducible $U(n)$ -submodule. Moreover, for $k, l \in \mathbb{N}$ we have $\mathcal{P}^k(\mathbb{C}^n) \ncong \mathcal{P}^l(\mathbb{C}^n)$ as $U(n)$-modules and $\mathcal{P}^k(\mathbb{C}^n) \perp \mathcal{P}^l(\mathbb{C}^n)$ whenever $k \neq l$.
In particular, for every $\alpha > -1$ we have

$$H^2_\alpha(\mathbb{B}^n) = \bigoplus_{k\in\mathbb{N}} \mathcal{P}^k(\mathbb{C}^n)$$

as an orthogonal direct sum of Hilbert spaces that yields the decomposition of $H^2_\alpha(\mathbb{B}^n)$ into irreducible $U(n)$-modules.

$U(n)$-intertwining operators

Theorem

For every $\alpha > -1$, the algebra $End_{U(n)}(H^2_\alpha(\mathbb{B}^n))$ is commutative. More precisely, every operator $T \in End_{U(n)}(H^2_\alpha(\mathbb{B}^n))$ is unitarily equivalent to RTR^* which is the multiplication operator on $l^2(\mathbb{Z}^n_+)$ by the function

$$\gamma_T : \mathbb{Z}^n_+ \longrightarrow \mathbb{C}$$
$$\gamma_T(m) = \langle Te_m, e_m \rangle$$

Furthermore, let us choose for every $k \in \mathbb{N}$ a unitary vector $u_k \in P^k(\mathbb{C}^n)$ consider the function

$$\hat{\gamma}_T : \mathbb{N} \longrightarrow \mathbb{C}$$
$$\hat{\gamma}_T(k) = \langle Tu_k, u_k \rangle$$

Then, we have $\hat{\gamma}_T(|m|) = \gamma_T(m)$ for every $m \in N$. Moreover, $\gamma_T(m) = \gamma_T(m')$ whenever $|m| = |m'|$.

Toeplitz operator with radial symbol

Theorem

The C^*-algebra generated by Toeplitz operators with radial symbols is commutative.
In other word, it is well known that the Toeplitz operator with radial symbols is unitary equivalent to

$$RTR^* = \gamma_a(m)I$$

action on $l_2(\mathbb{Z}_+^n)$ where

$$\gamma_a(m) = \gamma_a(m')$$

for all $|m| = |m'|$.

Contents

We define the set of indices

$$\mathcal{I} = \{\iota = (\iota_1, \ldots, \iota_n) \in \mathbb{Z}_+^n : \iota_1 \geq \iota_2 \geq \cdots \geq \iota_n \geq 0\}.$$

For each $\iota \in \mathcal{I}$, we also define the set

$$\mathcal{I}_\iota = \{\sigma(\iota) = (\iota_{\sigma(1)}, \ldots, \iota_{\sigma(n)}) \in \mathbb{Z}_+^n : \sigma \in S_n\}.$$

Let us denote by $\mathcal{P}_\iota(\mathbb{C}^n)$ the space of polynomials generated by the monomials of the form

$$z^{\sigma(\iota)} = z_1^{\iota_{\sigma(1)}} \cdots z_n^{\iota_{\sigma(n)}}$$

where $\sigma \in S_n$.

$S_n \rtimes \mathbb{T}^n$-intertwining operators

Proposition

The decomposition of $\mathcal{P}(\mathbb{C}^n)$ into irreducible $S_n \rtimes \mathbb{T}^n$-modules is given by

$$\mathcal{P}(\mathbb{C}^n) = \bigoplus_{\iota \in \mathcal{I}} \mathcal{P}_\iota(\mathbb{C}^n)$$

More precisely, for every $\iota \in \mathcal{I}$, the space $\mathcal{P}_\iota(\mathbb{C}^n)$ is an irreducible $S_n \rtimes \mathbb{T}^n$-submodule. Moreover, for $\iota_1, \iota_2 \in \mathcal{I}$ we have $\mathcal{P}_{\iota_1}(\mathbb{C}^n) \not\cong \mathcal{P}_{\iota_2}(\mathbb{C}^n)$ as $S_n \rtimes \mathbb{T}^n$-modules and $\mathcal{P}_{\iota_1}(\mathbb{C}^n) \perp \mathcal{P}_{\iota_2}(\mathbb{C}^n)$ whenever $\iota_1 \neq \iota_2$.
In particular, for every $\alpha > -1$ we have

$$H^2_\alpha(\mathbb{B}^n) = \bigoplus_{\iota \in \mathcal{I}} \mathcal{P}_\iota(\mathbb{C}^n)$$

as an orthogonal direct sum of Hilbert spaces that yields the decomposition of $H^2_\alpha(\mathbb{B}^n)$ into irreducible $S_n \rtimes \mathbb{T}^n$-modules.

Note that for each monomial z^m with $m \in \mathbb{Z}_+^n$ the action

$$
\begin{aligned}
(\sigma, t) \cdot z^m = (\sigma^{-1}(t^{-1}z))^m &= (t^{-1}_{\sigma^{-1}(1)} z_{\sigma^{-1}(1)})^{m_1} \cdots (t^{-1}_{\sigma^{-1}(n)} z_{\sigma^{-1}(n)})^{m_n} \qquad (1) \\
&= t^{-m_1}_{\sigma^{-1}(1)} \cdots t^{-m_n}_{\sigma^{-1}(n)} z^{m_1}_{\sigma^{-1}(1)} \cdots z^{m_n}_{\sigma^{-1}(n)} \\
&= t_1^{-m_{\sigma(1)}} \cdots t_n^{-m_{\sigma(n)}} z_1^{m_{\sigma(1)}} \cdots z_n^{m_{\sigma(n)}} \\
&= t^{-\sigma(m)} z^{\sigma(m)}
\end{aligned}
$$

where $\sigma(m) = (m_{\sigma(1)} \ldots m_{\sigma(n)}) \in \mathbb{Z}_+^n$.

$S_n \rtimes \mathbb{T}^n$-intertwining operators

Theorem

For every $\alpha > -1$, the algebra $End_{S_n \rtimes \mathbb{T}^n}(H^2_\alpha(\mathbb{B}^n))$ is commutative. More precisely, every operator $T \in End_{S_n \rtimes \mathbb{T}^n}(H^2_\alpha(\mathbb{B}^n))$ is unitarily equivalent to RTR^* which is the multiplication operator on $l^2(\mathbb{Z}^n_+)$ by the function

$$\gamma_T : \mathbb{Z}^n_+ \longrightarrow \mathbb{C} \text{with } \gamma_T(m) = \langle Te_m, e_m \rangle$$

Furthermore, let us choose for every $\iota \in \mathcal{I}$ a unitary vector $u_\iota \in P_\iota(\mathbb{C}^n)$ consider the function

$$\hat{\gamma}_T : \mathcal{I} \longrightarrow \mathbb{C}$$
$$\hat{\gamma}_T(\iota) = \langle Tu_\iota, u_\iota \rangle$$

In particular, we can consider $u_\iota = e_\iota(z) = \sqrt{\frac{\Gamma(n+|\iota|+\alpha+1)}{\iota!\Gamma(n+\alpha+1)}} z^\iota$. Then, we have $\hat{\gamma}_T(\iota) = \gamma_T(m)$ for every $m \in \mathcal{I}_\iota$. Moreover, $\gamma_T(m) = \gamma_T(m')$ whenever $m, m' \in \mathcal{I}_\iota$.

Corollary

With the above notation, the assignment

$$T \longmapsto \hat{\gamma}_T$$

defines an isomorphism of algebras

$$End_{S_n \rtimes \mathbb{T}^n}(H^2_\alpha(\mathbb{B}^n)) \longrightarrow l^\infty(\mathcal{I})$$

We define a symmetric separately radial function $a \in L^\infty(\mathbb{B}^n)$ if can be written as follows

$$a(z) = a(|z_1|^2, \dots, |z_n|^2) \tag{2}$$

and satisfies that

$$a(\sigma(z)) = a(z) \text{ where } \sigma \in S_n \tag{3}$$

for almost every $z \in \mathbb{B}^n$.
In other words, the function a is symmetric separately radial if and only if that it is $S_n \rtimes \mathbb{T}^n$-invariant.

Theorem

The C^*-algebra generated by Toeplitz operators with symmetric separately radial symbols is commutative, i. e. , for every $\alpha > -1$ the Toeplitz operator is unitary equivalent to

$$RT_aR^* = \bigoplus_{\iota\in\mathcal{I}} \gamma_a(\iota) I|_{l_2(\mathcal{I}_\iota)} \text{ action on } l_2(\mathbb{Z}_+^n) = \bigoplus_{\iota\in\mathcal{I}} l_2(\mathcal{I}_\iota),$$

and the spectral function is given by

$$\gamma_a(\iota) = \frac{2^n\Gamma(n+|\iota|+\alpha+1)}{\iota!\Gamma(n+\alpha+1)} \int_{\tau(\mathbb{B}^n)} a(r)r^{2\iota}(1-|r|^2)^\alpha \prod_{k=1}^n r_k d_k.$$

In other word, it is well known that the Toeplitz operator with symmetric separately radial symbols is unitary equivalent to $RTR^* = \gamma_a(m)I$ action on $l_2(\mathbb{Z}_+^n)$ where

$$\gamma_a(m) = \frac{\Gamma(n+|m|+\alpha+1)}{m!\Gamma(n+\alpha+1)} \int_{\tau(\mathbb{B}^n)} a(r)r^{2m}(1-|r|^2)^\alpha \prod_{k=1}^n r_k d_k.$$

Then the spectral function satisfies that $\gamma_a(m) = \gamma_a(\iota)$ for all $m \in \mathcal{I}_\iota$.

Example

For $n = 2$, we consider the symmetric separately radial function

$$a_0(r_1^2, r_2^2) = r_1^2 r_2^2$$

The set the indices for the decomposition of the Bergman spaces $H^2_\alpha(\mathbb{B}^2)$ associated to the group $S_2 \rtimes \mathbb{T}^2$ are given by $\mathcal{I} = \{(k_1, k_2) : k_1 \geq k_2 \geq 0\}$. The subspace of the Bergman spaces $H^2_\alpha(\mathbb{B}^2)$ associated to $(k_1, k_2) \in \mathcal{I}$ is defined by

$$\mathcal{P}_{(k_1,k_2)}(\mathbb{B}^n) = \begin{cases} \langle z_1^{k_1} z_2^{k_1} \rangle & k_1 = k_2 \\ \langle z_1^{k_1} z_2^{k_2}, z_1^{k_2} z_2^{k_1} \rangle & k_1 \neq k_2 \end{cases}$$

where $\langle z_1^{k_1} z_2^{k_2}, z_1^{k_2} z_2^{k_1} \rangle$ is the space generated by these polynomials.

Example

Now, the spectral function of Toeplitz operator with symbol a_0 is given by

$$\begin{aligned}
&\gamma_{a_0}(k_1, k_2) \\
&= \frac{2^2\Gamma(n+k_1+k_2+\alpha+1)}{k_1!k_2!\Gamma(n+\alpha+1)} \int_{\tau(\mathbb{B}^2)} r_1^{2k_1+3} r_2^{2k_2+3}(1-r_1^2-r_2^2)^\alpha dr_1 dr_2 \\
&= \frac{(k_1+1)(k_2+1)}{(n+k_1+k_2+\alpha+2)(n+k_1+k_2+\alpha+1)}.
\end{aligned}$$

Note that if we consider $k_1, k_2, k_3 \in \mathbb{Z}_+$ with $k_1 > k_2$ and $k_1 + k_2 = 2k_3$ then

$$\gamma_{a_0}(k_1, k_2) \neq \gamma_{a_0}(k_3, k_3)$$

The above equation exemplify the difference between the spectral functions associated with symmetric separately radial and radial symbols respectively. Since a the spectral functions associated to a radial symbol is constant respect to quantity $k_1 + k_2$.

Contents

For each $\iota = (\iota_1, \dots \iota_n) \in \mathcal{I}$, we also define the subsets of $\mathcal{I}_\iota$ as follows

$$\mathcal{I}_\iota^+ = \{\sigma(\iota) = (\iota_{\sigma(1)}, \dots, \iota_{\sigma(n)}) \in \mathbb{Z}_+^n : \sigma \in A_n\}$$

and

$$\mathcal{I}_\iota^- = \{\sigma(\iota) = (\iota_{\sigma(1)}, \dots, \iota_{\sigma(n)}) \in \mathbb{Z}_+^n : \sigma \in S_n \setminus A_n\}$$

We can check the following relation of the above sets

$$\mathcal{I}_\iota^+ \cap \mathcal{I}_\iota^- = \emptyset \text{ for } \iota_1 > \cdots > \iota_n \tag{4}$$

$$\mathcal{I}_\iota^+ = \mathcal{I}_\iota^- \text{ otherwise} \tag{5}$$

Now, we introduce the subsets of $\mathcal{I}$ given by

$$\mathcal{I}^0 = \{\iota = (\iota_1, \dots, \iota_n) \in \mathcal{I} : \exists l_1, l_2 \in \mathbb{Z}^+ \text{ such that } \iota_{l_1} = \iota_{l_2}\} \tag{6}$$

$$\mathcal{I}^c = \{\iota = (\iota_1, \dots, \iota_n) \in \mathcal{I} : \iota_1 > \cdots > \iota_n\} \tag{7}$$

Let us denote by $\mathcal{P}_\iota^+(\mathbb{C}^n)$ and $\mathcal{P}_\iota^+(\mathbb{C}^n)$ the spaces of polynomials generated by all the monomials of the form

$$z^{\sigma(\iota)} = z_1^{\iota_{\sigma(1)}} \cdots z_n^{\iota_{\sigma(n)}}$$

where σ is even and odd respectively.
Using the above relations we can easily obtain the following

$$\mathcal{P}_\iota(\mathbb{C}^n) = \mathcal{P}_\iota^+(\mathbb{C}^n) = \mathcal{P}_\iota^-(\mathbb{C}^n) \text{ for } \iota \in \mathcal{I}^0 \tag{8}$$

$$\mathcal{P}_\iota^+(\mathbb{C}^n) \perp \mathcal{P}_\iota^-(\mathbb{C}^n) \text{ for } \iota \in \mathcal{I}^c \tag{9}$$

$A_n \rtimes \mathbb{T}^n$-intertwining operators

Proposition

The decomposition of $\mathcal{P}(\mathbb{C}^n)$ into irreducible $A_n \rtimes \mathbb{T}^n$-modules is given by

$$\mathcal{P}(\mathbb{C}^n) = \left(\bigoplus_{\iota \in \mathcal{I}^0} \mathcal{P}_\iota(\mathbb{C}^n) \right) \bigoplus \left(\bigoplus_{\iota \in \mathcal{I}^c} \mathcal{P}_\iota^+(\mathbb{C}^n) \bigoplus \mathcal{P}_\iota^-(\mathbb{C}^n) \right)$$

More precisely, we have the following statements

1. *For every $\iota \in \mathcal{I}^0$, the space $\mathcal{P}_\iota(\mathbb{C}^n)$ is an irreducible $A_n \rtimes \mathbb{T}^n$ -submodule.*
2. *For every $\iota \in \mathcal{I}^c$, the spaces $\mathcal{P}_\iota^+(\mathbb{C}^n)$ and $\mathcal{P}_\iota^-(\mathbb{C}^n)$ are irreducible $A_n \rtimes \mathbb{T}^n$-submodule.*

Moreover, we obtain the following list of relations

$A_n \rtimes \mathbb{T}^n$-intertwining operators

Proposition (Continuation...)

1. *For* $\iota_1, \iota_2 \in \mathcal{I}$, *we have* $\mathcal{P}^{\pm}_{\iota_1}(\mathbb{C}^n) \ncong \mathcal{P}^{\pm}_{\iota_2}(\mathbb{C}^n)$ *as* $A_n \rtimes \mathbb{T}^n$*-modules and* $\mathcal{P}^{\pm}_{\iota_1}(\mathbb{C}^n) \perp \mathcal{P}^{\pm}_{\iota_2}(\mathbb{C}^n)$ *whenever* $\iota_1 \neq \iota_2$
2. *For* $\iota \in \mathcal{I}^c$, *we have* $\mathcal{P}^{+}_{\iota_1}(\mathbb{C}^n) \ncong \mathcal{P}^{-}_{\iota_2}(\mathbb{C}^n)$ *as* $A_n \rtimes \mathbb{T}^n$*-modules and* $\mathcal{P}^{+}_{\iota}(\mathbb{C}^n) \perp \mathcal{P}^{-}_{\iota}(\mathbb{C}^n)$

In particular, for every $\alpha > -1$ *we have*

$$H^2_\alpha(\mathbb{B}^n) = \left(\bigoplus_{\iota \in \mathcal{I}^0} \mathcal{P}_\iota(\mathbb{C}^n) \right) \bigoplus \left(\bigoplus_{\iota \in \mathcal{I}^c} \mathcal{P}^{+}_{\iota}(\mathbb{C}^n) \bigoplus \mathcal{P}^{-}_{\iota}(\mathbb{C}^n) \right)$$

as an orthogonal direct sum of Hilbert spaces that yields the decomposition of $H^2_\alpha(\mathbb{B}^n)$ *into irreducible* $A_n \rtimes \mathbb{T}^n$*-modules.*

$A_n \rtimes \mathbb{T}^n$-intertwining operators

Theorem

For every $\alpha > -1$, the algebra $End_{A_n \rtimes \mathbb{T}^n}(H^2_\alpha(\mathbb{B}^n))$ is commutative. More precisely, every operator $T \in End_{A_n \rtimes \mathbb{T}^n}(H^2_\alpha(\mathbb{B}^n))$ is unitarily equivalent to $RTR^* = \gamma_a I$ which is the multiplication operator on $l^2(\mathbb{Z}^n_+)$ by the function Furthermore, let us choose the following unitary vectors

1. *For every $\iota \in \mathcal{I}^0$ a unitary vector $u_\iota = e_\iota(z) \in P_\iota(\mathbb{B}^n)$.*
2. *For every $\iota \in \mathcal{I}^c$ a unitary vector $u_\iota^+ = e_\iota(z) \in P_\iota^+(\mathbb{B}^n)$.*
3. *For every $\iota \in \mathcal{I}^c$ a unitary vector $u_\iota^- = e_{\sigma(\iota)}(z) \in P_\iota^-(\mathbb{B}^n)$ where sigma is odd.*

$A_n \rtimes \mathbb{T}^n$-intertwining operators

Theorem (Continuation...)

We consider the function

$$\hat{\gamma}_T : \mathcal{I}^0 \sqcup \mathcal{I}_+^c \sqcup \mathcal{I}_-^c \longrightarrow \mathbb{C}$$

$$\hat{\gamma}_T(\iota) = \begin{cases} \langle Tu_\iota, u_\iota \rangle & \text{for } \iota \in \mathcal{I}^0 \\ \langle Tu_\iota^+, u_\iota^+ \rangle & \text{for } \iota \in \mathcal{I}_+^c = \mathcal{I}^c \\ \langle Tu_\iota^-, u_\iota^- \rangle & \text{for } \iota \in \mathcal{I}_-^c = \mathcal{I}^c \end{cases}$$

Then,

1. *For $\iota \in \mathcal{I}^0$ we have $\hat{\gamma}_T(\iota) = \gamma_T(m)$ for every $m \in \mathcal{I}_\iota$, i.e., $\gamma_T(m) = \gamma_T(m')$ whenever $m, m' \in \mathcal{I}_\iota$.*
2. *For $\iota \in \mathcal{I}^c$ we have $\hat{\gamma}_T(\iota) = \gamma_T(m)$ for every $m \in \mathcal{I}_\iota^+$, i.e., $\gamma_T(m) = \gamma_T(m')$ whenever $m, m' \in \mathcal{I}_\iota^+$.*
3. *Consider σ an odd permutation. For $\iota \in \mathcal{I}^c$ we have $\hat{\gamma}_T(\sigma(\iota)) = \gamma_T(m)$ for every $m \in \mathcal{I}_\iota^-$, i.e., $\gamma_T(m) = \gamma_T(m')$ whenever $m, m' \in \mathcal{I}_\iota^-$.*

We define a alternate separately radial function $a \in L^\infty(\mathbb{B}^n)$ if can be written as follows

$$a(z) = a(|z_1|^2, \dots, |z_n|^2) \tag{10}$$

and satisfies that

$$a(\sigma(z)) = a(z) \text{ where } \sigma \in A_n \tag{11}$$

for almost every $z \in \mathbb{B}^n$.
In other words, the function a is alternate separately radial if and only if that it is $A_n \rtimes \mathbb{T}^n$-invariant.

Theorem

The C^*-algebra generated by Toeplitz operators with alternate separately radial symbols is commutative, i. e. , for every $\alpha > -1$ the Toeplitz operator is unitary equivalent to

$$RT_aR^* = \left(\bigoplus_{\iota\in\mathcal{I}_0}\gamma_a(\iota)I|_{l_2(\mathcal{I}_\iota)}\right)\bigoplus\left(\bigoplus_{\iota\in\mathcal{I}^c}\gamma_a(\iota)I|_{l_2(\mathcal{I}_\iota^+)}\oplus\gamma_a(\sigma(\iota))I|_{l_2(\mathcal{I}_\iota^-)}\right),$$

action on

$$l_2(\mathbb{Z}_+^n) = \left(\bigoplus_{\iota\in\mathcal{I}_0} l_2(\mathcal{I}_\iota)\right)\bigoplus\left(\bigoplus_{\iota\in\mathcal{I}^c} l_2(\mathcal{I}_\iota^+)\oplus l_2(\mathcal{I}_\iota^-)\right)$$

where σ is odd and the spectral function is given by

$$\gamma_a(\iota) = \frac{2^n\Gamma(n+|\iota|+\alpha+1)}{\iota!\Gamma(n+\alpha+1)}\int_{\tau(\mathbb{B}^n)} a(r)r^{2\iota}(1-|r|^2)^\alpha\prod_{k=1}^n r_k d_k.$$

and

$$\gamma_a(\sigma(\iota)) = \frac{2^n\Gamma(n+|\iota|+\alpha+1)}{\iota!\Gamma(n+\alpha+1)}\int_{\tau(\mathbb{B}^n)} a(r)r^{2\sigma(\iota)}(1-|r|^2)^\alpha\prod_{k=1}^n r_k d_k.$$

Theorem (Continuation...)

In other word, it is well known that the Toeplitz operator with alternate separately radial symbols is unitary equivalent to $RT_aR^* = \gamma_a(m)I$ action on $l_2(\mathbb{Z}^n_+)$ where

$$\gamma_a(m) = \frac{\Gamma(n+|m|+\alpha+1)}{m!\Gamma(n+\alpha+1)} \int_{\tau(\mathbb{B}^n)} a(r)r^{2m}(1-|r|^2)^\alpha \prod_{k=1}^{n} r_k d_k.$$

Then the spectral function satisfies that

1. *For $\iota \in \mathcal{I}^0$ we have $\gamma_a(\iota) = \gamma_a(m)$ for every $m \in \mathcal{I}_\iota$, i.e., $\gamma_a(m) = \gamma_a(m')$ whenever $m, m' \in \mathcal{I}_\iota$.*
2. *For $\iota \in \mathcal{I}^c$ we have $\gamma_a(\iota) = \gamma_a(m)$ for every $m \in \mathcal{I}^+_\iota$, i.e., $\gamma_a(m) = \gamma_a(m')$ whenever $m, m' \in \mathcal{I}^+_\iota$.*
3. *Consider σ an odd permutation. For $\iota \in \mathcal{I}^c$ we have $\gamma_a(\sigma(\iota)) = \gamma_a(m)$ for every $m \in \mathcal{I}^-_\iota$, i.e., $\gamma_a(m) = \gamma_a(m')$ whenever $m, m' \in \mathcal{I}^-_\iota$.*

Contents

We define a antisymmetric separately radial function $a \in L^\infty(\mathbb{B}^n)$ if can be written as follows

$$a(z) = a(|z_1|, \ldots, |z_n|) \tag{12}$$

and satisfies that

$$a(\sigma(z)) = \mathrm{Sgn}(\sigma)a(z) \text{ where } \sigma \in S_n \tag{13}$$

for almost every $z \in \mathbb{B}^n$.
In particular, every antisymmetric separately radial function is alternate separately radial.

Theorem

The C^*-algebra generated by Toeplitz operators with antisymmetric separately radial symbols is commutative, i. e. , for every $\alpha > -1$ the Toeplitz operator is unitary equivalent to

$$RT_aR^* = \left(\bigoplus_{\iota\in\mathcal{I}_0} \gamma_a(\iota) I|_{l_2(\mathcal{I}_\iota)}\right) \bigoplus \left(\bigoplus_{\iota\in\mathcal{I}^c} \gamma_a(\iota) I|_{l_2(\mathcal{I}_\iota^+)} \oplus \gamma_a(\sigma(\iota)) I|_{l_2(\mathcal{I}_\iota^-)}\right),$$

action on

$$l_2(\mathbb{Z}_+^n) = \left(\bigoplus_{\iota\in\mathcal{I}_0} l_2(\mathcal{I}_\iota)\right) \bigoplus \left(\bigoplus_{\iota\in\mathcal{I}^c} l_2(\mathcal{I}_\iota^+) \oplus l_2(\mathcal{I}_\iota^-)\right)$$

where σ is odd and the spectral function satisfies that

1. *For $\iota \in \mathcal{I}^0$ we have $\gamma_a(\iota) = 0$.*
2. *For $\iota \in \mathcal{I}^c$ we have $\gamma_a(\sigma(\iota)) = -\gamma_a(\iota)$*

Theorem (Continuation...)

In other word, it is well known that the Toeplitz operator with antisymmetric separately radial symbols is unitary equivalent to $RTR^* = \gamma_a(m)I$ *action on* $l_2(\mathbb{Z}_+^n)$ *where*

$$\gamma_a(m) = \frac{\Gamma(n+|m|+\alpha+1)}{m!\Gamma(n+\alpha+1)} \int_{\tau(\mathbb{B}^n)} a(r) r^{2m} (1-|r|^2)^\alpha \prod_{k=1}^{n} r_k d_k.$$

Then the spectral function satisfies that

1. *For* $\iota \in \mathcal{I}^0$ *we have* $\gamma_a(m) = 0$ *for every* $m \in \mathcal{I}_\iota$.
2. *For* $\iota \in \mathcal{I}^c$ *we have* $\gamma_a(\iota) = \gamma_a(m)$ *for every* $m \in \mathcal{I}_\iota^+$.
3. *Consider* σ *an odd permutation. For* $\iota \in \mathcal{I}^c$ *we have* $\gamma_a(m) = -\gamma_a(\iota)$ *for every* $m \in \mathcal{I}_\iota^-$.

Remark

1. *If a is an alternate separately radial function and σ odd permutation, then $a_\sigma(z) = a(\sigma(z))$ is an alternate separately radial function.*
2. *If a is an alternate separately radial function and σ, β are odd permutations, then $a(\sigma(z)) = a(\beta(z))$ and $a(\beta\sigma(z)) = a(z)$.*
3. *If a is an alternate separately radial function and σ odd permutation, then*
$$a^+ = \frac{a + a_\sigma}{2}$$
is a symmetric function.
4. *If a is an alternate separately radial function and σ odd permutation, then*
$$a^- = \frac{a - a_\sigma}{2}$$
is a antisymmetric function.
5. *If a is an alternate separately radial function and σ odd permutation, then $a = a^+ + a^-$.*

Toeplitz operator with alternate separately radial symbols $T_a = T_{a^+} + T_{a^-}$. These operators are unitary equivalent to

$$RT_{a^+}R^* = \left(\bigoplus_{\iota \in \mathcal{I}_0} \gamma_a(\iota) I|_{l_2(\mathcal{I}_\iota)} \right) \oplus \bigoplus_{\iota \in \mathcal{I}^c} \gamma_a(\iota) \left(I|_{l_2(\mathcal{I}_\iota^+)} \oplus I|_{l_2(\mathcal{I}_\iota^-)} \right),$$

and

$$RT_{a^-}R^* = \left(\bigoplus_{\iota \in \mathcal{I}_0} 0|_{l_2(\mathcal{I}_\iota)} \right) \oplus \bigoplus_{\iota \in \mathcal{I}^c} \gamma_a(\iota) \left(I|_{l_2(\mathcal{I}_\iota^+)} \oplus (-I)|_{l_2(\mathcal{I}_\iota^-)} \right).$$

Thank You

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