## Eigenvalue superposition expansion for Toeplitz matrix-sequences, generated by linear combinations of matrix-order dependent symbols

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## FE vs IgA: Tom Hughes 2005



## FE vs IgA: Tom Hughes 2005



## RTCP

The focus concerns the case where $f$ is real-valued with a cosine expansion, that is, a function of the form

$$
f(\theta)=\hat{f}_{0}+2 \sum_{k=1}^{m} \hat{f}_{k} \cos (k \theta), \quad \hat{f}_{0}, \hat{f}_{1}, \ldots, \hat{f}_{m} \in \mathbb{R},
$$

so that $f(2 \pi-s)=f(s), s \in[0, \pi]$.

## RTCP

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$$

so that $f(2 \pi-s)=f(s), s \in[0, \pi]$. When approximating the operator $(-1)^{q} \frac{\mathrm{~d}^{2 q}}{\mathrm{~d} x^{2 q}}, q=0,1,2, \ldots$, we end up with structures as $T_{n}\left(f_{q}\right)$ with $f_{q}$ being a monotone, real-valued cosine polynomial of the form

$$
f_{q}(\theta)=(2-2 \cos (\theta))^{q}, \quad q=0,1,2, \ldots,
$$

(minimal bandwidth centered Finite Differences of order 2).

## RTCP

When using the IgA with maximal regularity, we will obtain structures like $T_{n}\left(g_{q}\right)$ with $g_{q}$ being a real-valued cosine polynomial of the form

$$
g_{q}(\theta)=(2-2 \cos (\theta))^{q} p_{q}(\theta), \quad q=0,1,2, \ldots,
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where $p_{q}$ is a strictly positive cosine polynomial.

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where $p_{q}$ is a strictly positive cosine polynomial.

- The considered Finite Differences are characterized by $O\left(n^{-2}\right)$ precision and minimal bandwidth, while for $g_{q}$ the bandwidth is larger, but the precision order is much higher.
- The fractional analysis approach will produce a Toeplitz matrix with a symbol like $g_{q}$ with $q \in \mathbb{Q}$ this time.


## RTCP

The $n$th Toeplitz matrix generated by $f$ is the real symmetric matrix given by

$$
T_{n}(f)=\left[\begin{array}{ccccccccccc}
\hat{f}_{0} & \hat{f}_{1} & \cdots & \hat{f}_{m} & & & & & & & \\
\hat{f}_{1} & \ddots & \ddots & & \ddots & & & & & & \\
\vdots & \ddots & \ddots & \ddots & & \ddots & & & & & \\
\hat{f}_{m} & & \ddots & \ddots & \ddots & & \ddots & & & & \\
& \ddots & & \ddots & \ddots & \ddots & & \ddots & & & \\
& & \hat{f}_{m} & \cdots & \hat{f}_{1} & \hat{f}_{0} & \hat{f}_{1} & \cdots & \hat{f}_{m} & & \\
& & & \ddots & & \ddots & \ddots & \ddots & & \ddots & \\
& & & & \ddots & & \ddots & \ddots & \ddots & & \hat{f}_{m} \\
& & & & & \ddots & & \ddots & \ddots & \ddots & \vdots \\
& & & & & & \ddots & & \ddots & \ddots & \hat{f}_{1} \\
& & & & & & & \hat{f}_{m} & \cdots & \hat{f}_{1} & \hat{f}_{0}
\end{array}\right],
$$

our analysis will be not restricted to banded matrices.

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## The Problem and the Literature

Under the simple loop condition over the interval $[-\pi, \pi]$ we find

$$
\lambda_{j}\left(T_{n}(f)\right)=f\left(\sigma_{j, n}\right)+c_{1}\left(\sigma_{j, n}\right) h+c_{2}\left(\sigma_{j, n}\right) h^{2}+\cdots+O\left(h^{\alpha}\right),
$$

where $h=\frac{1}{n+1}, \sigma_{j, n}=\pi j h$, and $c_{k}, \gamma_{k}$ are some bounded coefficients depending only on $f, \alpha$ is a positive integer depending on the smoothness of $f$.

## The Problem and the Literature

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where $h=\frac{1}{n+1}, \sigma_{j, n}=\pi j h$, and $c_{k}, \gamma_{k}$ are some bounded coefficients depending only on $f, \alpha$ is a positive integer depending on the smoothness of $f$.

The numerical results presented in the literature suggests that the effective conditions for the expansion to hold are weaker: an even character of $f$ and monotonicity over $[0, \pi]$.

## The Problem and the Literature

We investigate the superposition caused over this expansion, when considering a linear combination of symbols, that is

$$
\lambda_{j}\left(T_{n}\left(f_{0}+\beta_{n}^{(1)} f_{1}+\beta_{n}^{(2)} f_{2}\right)\right),
$$

where the symbols $f_{j}$ are either simple loop or satisfy the weaker conditions mentioned before.

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$$

where the symbols $f_{j}$ are either simple loop or satisfy the weaker conditions mentioned before.

- we formally prove that the asymptotic expansion holds also in this setting under mild assumptions;
- there is much more to investigate, opening the door to linear in time algorithms for the computation of eigenvalues of large matrices of this type.


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## GLT Theory

Let $f: D \rightarrow \mathbb{C}$ be a measurable function defined on the Lebesgue measurable set $D$ of positive and finite measure. Assume that $\left\{A_{n}\right\}_{n}$ is a sequence of matrices such that $\operatorname{dim}\left(A_{n}\right)=d_{n} \rightarrow \infty$, as $n \rightarrow \infty$ and with eigenvalues $\lambda_{j}\left(A_{n}\right)$ and singular values $\sigma_{j}\left(A_{n}\right), j=1, \ldots, d_{n}$.

## GLT Theory

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We say that $\left\{A_{n}\right\}_{n}$ is distributed as $f$ over $D$ in the sense of the eigenvalues, and we write $\left\{A_{n}\right\}_{n} \sim_{\lambda}(f, D)$, if

$$
\lim _{n \rightarrow \infty} \frac{1}{d_{n}} \sum_{j=1}^{d_{n}} F\left(\lambda_{j}\left(A_{n}\right)\right)=\frac{1}{\mu(D)} \int_{D} F(f(t)) \mathrm{d} t
$$

for every continuous function $F$ with compact support. In this case, we say that $f$ is the spectral symbol of $\left\{A_{n}\right\}_{n}$.

## GLT Theory

We say that $\left\{A_{n}\right\}_{n}$ is distributed as $f$ over $D$ in the sense of the singular values, and we write $\left\{A_{n}\right\}_{n} \sim_{\sigma}(f, D)$, if

$$
\lim _{n \rightarrow \infty} \frac{1}{d_{n}} \sum_{j=1}^{d_{n}} F\left(\sigma_{j}\left(A_{n}\right)\right)=\frac{1}{\mu(D)} \int_{D} F(|f(t)|) \mathrm{d} t
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## Theorem (Tilli, Tyrtyshnikov, Zamarashkin)

Let $f \in L^{1}([-\pi, \pi])$, then $\left\{T_{n}(f)\right\}_{n} \sim_{\sigma}(f,[-\pi, \pi])$. If $f$ is a real-valued function almost everywhere, then $\left\{T_{n}(f)\right\}_{n} \sim_{\lambda}(f,[-\pi, \pi])$.

## Why this linear combination?

## Theorem

For $F_{n} \equiv f_{0}+\beta_{n}^{(1)} f_{1}+\beta_{n}^{(2)} f_{2}$ and $X_{n} \equiv T_{n}\left(F_{n}\right)$ we have $\left\{X_{n}\right\}_{n} \sim_{\lambda} F$, where $F \equiv \lim _{n \rightarrow \infty} F_{n}$.

Here by making reference to the approximations by Finite Differences, for a fixed positive integer $l$, we could consider operators of the form

$$
\sum_{s=0}^{l}(-1)^{s} \alpha_{s} \frac{\mathrm{~d}^{2 s}}{\mathrm{~d} x^{2 s}}
$$

which, by linearity of the approximation technique of the involved operators, give raise to Toeplitz structures with the expression

$$
T_{n}\left(\sum_{s=0}^{l} \alpha_{s} h^{2(l-s)} f_{s}\right), \quad f_{s}(\theta)=(2-2 \cos (\theta))^{s} .
$$

## Why this linear combination?

In perfect analogy, in the case where the approximation is obtained via IgA, for a fixed positive integer $l$, we reach

$$
T_{n}\left(\sum_{s=0}^{l} \alpha_{s} h^{2(l-s)} g_{s}\right)
$$

## Why this linear combination?

In perfect analogy, in the case where the approximation is obtained via IgA, for a fixed positive integer $l$, we reach

$$
T_{n}\left(\sum_{s=0}^{l} \alpha_{s} h^{2(l-s)} g_{s}\right)
$$

In both cases it is evident that the related matrix-sequences have

$$
\sum_{s=0}^{l} \alpha_{s} h^{2(l-s)} F_{s}
$$

as GLT momentary symbols, with $F_{s}$ being either $f_{s}$ or $g_{s}$. In both cases we are interested in using the related symbols, and the superposition effect for computing

$$
\lambda_{j}\left(T_{n}\left(\sum_{s=0}^{l} \alpha_{s} h^{2(l-s)} F_{s}\right)\right),
$$

which is a special instance of our general problem.

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## The SL Approach

For a constant $\alpha \geqslant 0$, the well-known weighted Wiener algebra $W^{\alpha}$ is the collection of all functions $f:[0,2 \pi] \rightarrow \mathbb{C}$ whose Fourier coefficients $\hat{f}_{j}$ satisfy

$$
\|f\|_{\alpha} \equiv \sum_{j=-\infty}^{\infty}\left|\hat{f}_{j}\right|(|j|+1)^{\alpha}<\infty .
$$

We address real-valued symbols $f$ in $W^{\alpha}$, tracing out a simple loop and satisfying the following conditions:
(I) The range of $f$ is a segment $[0, \mu]$ with $\mu>0$.
(II) $f(0)=f(2 \pi)=0$ and $f^{\prime \prime}(0)=f^{\prime \prime}(2 \pi)>0$.
(III) There is a $\sigma_{0} \in(0,2 \pi)$ such that $f\left(\sigma_{0}\right)=\mu, f^{\prime}(\sigma)>0$ for $0<\sigma<\sigma_{0}$, $f^{\prime}(\sigma)<0$ for $\sigma_{0}<\sigma<2 \pi, f^{\prime}\left(\sigma_{0}\right)=0$, and $f^{\prime \prime}\left(\sigma_{0}\right)<0$.

The SL Approach


A typical simple-loop simbol.

## The SL Approach

According to the simple-loop method, by considering

$$
b_{f}(\sigma, s) \equiv \frac{f(\sigma)-f(s)}{2(\cos (s)-\cos (\sigma))} \quad(\sigma \in[0,2 \pi], s \in[0, \pi]),
$$

we obtain a real and continuous function, which is also bounded away from zero.

## The SL Approach

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b_{f}(\sigma, s) \equiv \frac{f(\sigma)-f(s)}{2(\cos (s)-\cos (\sigma))} \quad(\sigma \in[0,2 \pi], s \in[0, \pi]),
$$

we obtain a real and continuous function, which is also bounded away from zero.

The resulting operator $T\left(b_{f}(\cdot, s)\right)$ is invertible and therefore, since the finite section method can be applied, the related finite Toeplitz matrices $T_{n}\left(b_{f}(\cdot, s)\right)$ are also invertible.

Note that $b_{f}$ can be thought of as the quotient between $f-\lambda$ and $2(\cos (s)-\cos (\sigma))$, which is similar to the preconditioning process of the ill-conditioned matrix $T_{n}(f-\lambda)$.

## The SL Approach

We define the function $\eta_{f}:[0, \pi] \rightarrow \mathbb{R}$ by

$$
\begin{aligned}
\eta_{f}(s) & \equiv \frac{1}{4 \pi} \oint_{0}^{2 \pi} \frac{\log b_{f}(\sigma, s)}{\tan \left(\frac{\sigma-s}{2}\right)} \mathrm{d} \sigma-\frac{1}{4 \pi} \int_{0}^{2 \pi} \frac{\log b_{f}(\sigma, s)}{\tan \left(\frac{\sigma+s}{2}\right)} \mathrm{d} \sigma \\
& =\frac{\sin (s)}{2 \pi} \int_{0}^{2 \pi} \frac{\log b_{f}(\sigma, s)}{\cos (s)-\cos (\sigma)} \mathrm{d} \sigma .
\end{aligned}
$$

Now the eigenvalues of $T_{n}(f)$ are given by

$$
\lambda_{j}\left(T_{n}(f)\right)=f\left(\sigma_{j, n}\right)+\sum_{\ell=1}^{\lfloor\alpha\rfloor} c_{\ell}\left(\sigma_{j, n}\right) h^{\ell}+E_{j, n, \alpha} .
$$

## The SL Approach

$$
\lambda_{j}\left(T_{n}(f)\right)=f\left(\sigma_{j, n}\right)+\sum_{\ell=1}^{\lfloor\alpha\rfloor} c_{\ell}\left(\sigma_{j, n}\right) h^{\ell}+E_{j, n, \alpha}
$$

Where the following conditions are satisfied
(I) the eigenvalues of $T_{n}(f)$ are arranged in nondecreasing order;
(II) $h \equiv \frac{1}{n+1}$ and $\sigma_{j, n} \equiv \pi j h$;
(III) the coefficients $c_{\ell}$ depend only on $f$ and can be found explicitly, for example

$$
c_{1}=-f^{\prime} \eta_{f}, \quad c_{2}=\frac{1}{2} f^{\prime \prime} \eta_{f}^{2}+f^{\prime} \eta_{f} \eta_{f}^{\prime} ;
$$

(IV) $E_{j, n, \alpha}=O\left(h^{\alpha}\right)$ is the remainder (error) term, which satisfies the bounding $\left|E_{j, n, \alpha}\right| \leqslant c h^{\alpha}$ for some constant $c$ depending only on $f$.

## Our claims

For $f, g \in \mathrm{SL}^{\alpha}$ and a constant $\beta \in \mathbb{R}_{+}$, we investigate the relationship between the eigenvalues $\lambda_{j}\left(T_{n}(f)\right), \lambda_{j}\left(T_{n}(g)\right)$, and $\lambda_{j}\left(T_{n}(f+\beta g)\right)$. We easily obtain

$$
\lambda_{j}\left(T_{n}(f+\beta g)\right)=f\left(\sigma_{j, n}\right)+\beta g\left(\sigma_{j, n}\right)+O(h) .
$$

Indeed, as a challenge in the field, we are looking for a more detailed result involving a complete expansion and a real constant $\beta_{n}$ depending on $n$. The symbol $f+\beta_{n} g$ depends on $n$, as a consequence the actual simple-loop method can not be applied.

However, under proper adjustments, the quoted technique can be used when $\beta_{n}$ is $h$ or $h^{h}$.

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## The Symbol $f+g h$

## Theorem

Let $f$ and $g$ be two symmetric symbols in $\mathrm{SL}^{\alpha}$ with $\alpha \geqslant 2$. Then

$$
\lambda_{j}\left(T_{n}(f+g h)\right)=f\left(\sigma_{j, n}\right)+\sum_{\ell=1}^{\lfloor\alpha\rfloor} \Psi_{\ell}\left(\sigma_{j, n}\right) h^{\ell}+E_{j, n, \alpha},
$$

where $\Psi_{\ell}$ are bounded functions from $[0, \pi]$ to $\mathbb{R}$ depending only on $f, g$ that can be obtained explicitly.

The remainder $E_{j, n, \alpha}=O\left(h^{\alpha}\right)$ satisfies the bounding $\left|E_{j, n, \alpha}\right| \leqslant c h^{\alpha}$ for some constant $c$ depending only on $f$ and $g$.

## The Symbol $f+g h$

Where

$$
\begin{aligned}
& \Psi_{1}=g-f^{\prime} \eta_{f} \\
& \Psi_{2}=\frac{1}{2} f^{\prime \prime} \eta_{f}^{2}+f^{\prime} \eta_{f} \eta_{f}^{\prime}-f^{\prime} \psi-g^{\prime} \eta_{f}
\end{aligned}
$$

the function $\psi$ is given by the following singular integral

$$
\psi(s) \equiv \frac{\sin (s)}{2 \pi} \int_{0}^{2 \pi} \frac{b_{g}(\sigma, s)}{b_{f}(\sigma, s)(\cos (s)-\cos (\sigma))} \mathrm{d} \sigma .
$$

For the next result we will study the eigenvalues corresponding to the symbol $f+g h^{h}$, that is $\beta_{n}=h^{h}$. Consider the function

$$
\varphi(s) \equiv \frac{\sin (s)}{2 \pi} \int_{0}^{2 \pi} \frac{b_{g}(\sigma, s)}{\left(b_{f}(\sigma, s)+b_{g}(\sigma, s)\right)(\cos (s)-\cos (\sigma))} \mathrm{d} \sigma .
$$

## The Symbol $f+g h^{h}$

## Theorem

Let $f$ and $g$ be two symmetric symbols in $\mathrm{SL}^{\alpha}$ with $\alpha \geqslant 2$. Then

$$
\begin{aligned}
\lambda_{j}\left(T_{n}\left(f+g h^{h}\right)\right)= & f\left(\sigma_{j, n}\right)+g\left(\sigma_{j, n}\right) \\
& +\sum_{\ell=1}^{\lfloor\alpha\rfloor} \sum_{k=0}^{\ell} \Gamma_{\ell, k}\left(\sigma_{j, n}\right) h^{\ell} \log ^{k}(h)+E_{j, n, \alpha},
\end{aligned}
$$

where $\Gamma_{\ell, k}$ are bounded functions from $[0, \pi]$ to $\mathbb{R}$ depending only on $f, g$ that can be expressed explicitly. The remainder (error) term $E_{j, n, \alpha}=O\left(h^{\alpha}\left|\log ^{\alpha}(h)\right|\right)$ satisfies $\left|E_{j, n, \alpha}\right| \leqslant c h^{\alpha}\left|\log ^{\alpha}(h)\right|$ for some constant $c$ depending only on $f$ and $g$.

## Proofs ideas: The magic equation

Our specific aim is to recreate the so called simple-loop method working with the symbol $f+\beta_{n} g$ for two particular values of $\beta_{n}$, i.e. $h$ and $h^{h}$. As we will show, in both cases we arrive at the important equation

$$
(n+1) s+\eta_{f+\beta_{n} g}(s)=\pi j+\Delta_{j, n, \alpha},
$$

where $\Delta_{j, n, \alpha}$ satisfies some smoothness and bounding conditions.

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$$

where $\Delta_{j, n, \alpha}$ satisfies some smoothness and bounding conditions.
At this point we decided to expand $\eta_{f+\beta_{n} g}$ into factors with coefficients not involving $n$, then we made a technical work claiming the function on the left side is a contraction and henceforth, we can iterate in order to solve it for $s$.

## Proofs ideas: The expansion of $\eta$

Thus, we start with the following technical result.

## Lemma

As $n \rightarrow \infty$, we have
(i) $\eta_{f+g h}(s)=\eta_{f}(s)+\psi(s) h+O\left(h^{2}\right)$,
(ii) $\eta_{f+g h^{h}}(s)=\eta_{f+g}(s)+\varphi(s) h \log (h)+O\left(h^{2} \log ^{2}(h)\right)$,
where $\psi, \varphi$ are $\lfloor\alpha\rfloor-1$ times continuously differentiable functions, given by

$$
\begin{aligned}
& \psi(s) \equiv \frac{\sin (s)}{2 \pi} \int_{0}^{2 \pi} \frac{b_{g}(\sigma, s)}{b_{f}(\sigma, s)(\cos (s)-\cos (\sigma))} d \sigma, \\
& \varphi(s) \equiv \frac{\sin (s)}{2 \pi} \int_{0}^{2 \pi} \frac{b_{g}(\sigma, s)}{\left(b_{f}(\sigma, s)-b_{g}(\sigma, s)\right)(\cos (s)-\cos (\sigma))} d \sigma .
\end{aligned}
$$

## Proofs ideas: The simplest superposition

The following lemma shows how the eigenvalues of the sum of two Toeplitz matrices are related with the individual ones, which is highly nontrivial given the inherent nonlinearity of the eigenvalues with respect to the entries of the considered matrix.

## Lemma

Let $f$ and $g$ be two symmetric symbols in $\mathrm{SL}^{\alpha}$ with $\alpha \geqslant 2$. Then we have

$$
\begin{aligned}
\lambda_{j}\left(T_{n}(f+g)\right)= & \lambda_{j}\left(T_{n}(f)\right)+\lambda_{j}\left(T_{n}(g)\right) \\
& +\sum_{\ell=1}^{\lfloor\alpha\rfloor} Q_{\ell}\left(\sigma_{j, n}\right) h^{\ell}+E_{j, n, \alpha}
\end{aligned}
$$

where $Q_{\ell}$ are bounded functions from $[0, \pi]$ to $\mathbb{R}$ depending only on $f, g$, that can be expressed explicitly. The remainder $E_{j, n, \alpha}=O\left(h^{\alpha}\right)$ satisfies the bounding $\left|E_{j, n, \alpha}\right| \leqslant c h^{\alpha}$ for some constant $c$ depending only on $f$ and $g$.

## Proofs ideas: Basic iterative equation

## Lemma

For every sufficiently large natural number $n$ there exists a real-valued function $R_{n} \in C[0, \pi]$ with the following properties:
(i) a number $\lambda=f(s)+\beta_{n} g(s)$ is an eigenvalue of $T_{n}\left(f+\beta_{n} g\right)$ if and only if there is a $j \in \mathbb{Z}$ such that

$$
(n+1) s+\eta_{f+\beta_{n} g}(s)=j \pi+R_{n}(s) ;
$$

(ii) $R_{n}(0)=R_{n}(\pi)=0$;
(iii) $\left\|R_{n}\right\|_{L_{\infty}}=o\left(h^{\alpha-1}\right)$ as $n \rightarrow \infty$.

## Proofs ideas: Sketch of the case $\beta_{n}=h$

We reach

$$
G_{n}(s)=j \pi
$$

where $G_{n}(s) \equiv(n+1) s+\eta_{f}(s)+\psi(s) h-R_{n}(s)$. The function $G_{n}$ is continuous on the interval $[0, \pi]$. From SL theory know that
(i) the eigenvalues of $T_{n}(f+g h)$ are all distinct:

$$
\lambda_{1}\left(T_{n}(f+g h)\right)<\cdots<\lambda_{n}\left(T_{n}(f+g h)\right) ;
$$

(ii) the numbers $s_{j, n} \equiv[f+g h]_{[0, \pi]}^{-1}\left(\lambda_{j}\left(T_{n}(f+g h)\right)(j=1, \ldots, n)\right.$ satisfy the main relation with $R_{n}\left(s_{j, n}\right)=o\left(h^{\alpha-1}\right)$;
(iii) for every sufficiently large $n$, we have exactly one solution $s_{j, n} \in[0, \pi]$ for each $j=1, \ldots, n$.

## Proofs ideas: Sketch of the case $\beta_{n}=h$

Let $F_{n}(s) \equiv(n+1) s+\eta_{f}(s)+\psi(s) h$. Hence, $F_{n}(s)=\pi j$ has a unique solution $\hat{s}_{j, n}$ for each $j=1, \ldots, n$ satisfying the bounding $\left|s_{j, n}-\hat{s}_{j, n}\right|=o\left(h^{\alpha}\right)$, and that the function

$$
\Phi_{j, n}(s) \equiv \sigma_{j, n}-\eta_{f}(s) h-\psi(s) h^{2}
$$

is a contraction on $[0, \pi]$. Then by the Banach fixed point theorem, the sequence defined by

$$
\hat{s}_{j, n}^{(0)} \equiv \sigma_{j, n} \quad \text { and } \quad \hat{s}_{j, n}^{(\ell)} \equiv \Phi_{j, n}\left(\hat{s}_{j, n}^{(\ell-1)}\right) \quad(\ell \geqslant 1),
$$

satisfies $\left|\hat{s}_{j, n}-\hat{s}_{j, n}^{(\ell)}\right|=O\left(h^{\ell+1}\right)$.

## Proofs ideas: Sketch of the case $\beta_{n}=h$

We will iterate over the relation $s=\Phi_{j, n}(s)$. Obtaining

$$
\begin{aligned}
& \hat{s}_{j, n}^{(0)}=\sigma_{j, n}, \\
& \hat{s}_{j, n}^{(1)}=\sigma_{j, n}-\eta_{f}\left(\sigma_{j, n}\right) h+O\left(h^{2}\right), \\
& \hat{s}_{j, n}^{(2)}=\sigma_{j, n}-\eta_{f}\left(\sigma_{j, n}\right) h+\left\{\eta_{f}\left(\sigma_{j, n}\right) \eta_{f}^{\prime}\left(\sigma_{j, n}\right)-\psi\left(\sigma_{j, n}\right)\right\} h^{2}+O\left(h^{3}\right),
\end{aligned}
$$

which combined with $\left|s_{j, n}-\hat{s}_{j, n}^{(2)}\right|=o\left(h^{\alpha}\right)+O\left(h^{3}\right)$ gives us the result.

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Consider the simple-loop symbol given by

$$
f(\sigma)=\frac{(1+\rho)^{2}}{2} \cdot \frac{1-\cos (\sigma)}{1-2 \rho \cos (\sigma)+\rho^{2}} \quad(0 \leqslant \sigma \leqslant 2 \pi)
$$

for a constant $0<\rho<1$. The respective Fourier coefficients can be explicitly calculated as $\hat{f}_{k}=\frac{1}{4}\left(\rho^{2}-1\right) \rho^{|k|-1}$ for $k \neq 0$ and $\frac{1}{2}(1+\rho)$ for $k=0$. This symbol was inspired in the Kac-Murdock-Szegő Toeplitz matrices. We have

$$
\|f\|_{\alpha}=\frac{1+\rho}{2}+\frac{\rho^{2}-1}{2 \rho} \sum_{k=1}^{\infty} \rho^{k}(k+1)^{\alpha}
$$

which is finite for every $\alpha>0$. Then $f \in \mathrm{SL}^{\alpha}$ for any $\alpha>0$. In this case the function $\eta_{f}$ is therefore nicely given by

$$
\eta_{f}(s)=2 \arctan \left(\frac{\rho \sin (s)}{1-\rho \cos (s)}\right)
$$

Our second symbol is given by

$$
g(\sigma) \equiv 4 \sin ^{2}\left(\frac{\sigma}{2}\right)=2(1-\cos (\sigma)) .
$$

The respective Fourier coefficients can be calculated as $\hat{g}_{k}=-1$ for $k= \pm 1$, $\hat{g}_{k}=2$ for $k=0$, and $\hat{g}_{k}=0$ in any other case: we remind that the quoted symbol is related to the classical discrete Laplacian in one dimension. Hence $\|g\|_{\alpha}<\infty$ for any $\alpha>0$. Then $g \in \mathrm{SL}^{\alpha}$ for any $\alpha>0$. In this case we obtain $b_{g}(\sigma, s)=1$ and $\eta_{g}(s)=0$, and the eigenvalues of $T_{n}(g)$ can be obtained explicitly as $\lambda_{j}\left(T_{n}(g)\right)=g\left(\sigma_{j, n}\right)$.

## Numerical verifications: $f+g$

Since $W^{\alpha}$ is an algebra, the symbol $f+g$ clearly belongs to $\mathrm{SL}^{\alpha}$ for any $\alpha>0$. For $k=1,2,3$ let $\lambda_{j}^{(k)}\left(T_{n}(f)\right)$ be the $k$ th term approximation of $\lambda_{j}\left(T_{n}(f)\right)$ given by our formulas. Specifically we find

$$
\begin{aligned}
& \lambda_{j}^{(1)}\left(T_{n}(f+g)\right)=\lambda_{j}\left(T_{n}(f)\right)+\lambda_{j}\left(T_{n}(g)\right), \\
& \lambda_{j}^{(2)}\left(T_{n}(f+g)\right)=\lambda_{j}\left(T_{n}(f)\right)+\lambda_{j}\left(T_{n}(g)\right)+h Q_{1}\left(\sigma_{j, n}\right), \\
& \lambda_{j}^{(3)}\left(T_{n}(f+g)\right)=\lambda_{j}\left(T_{n}(f)\right)+\lambda_{j}\left(T_{n}(g)\right)+h Q_{1}\left(\sigma_{j, n}\right)+h^{2} Q_{2}\left(\sigma_{j, n}\right) .
\end{aligned}
$$

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& \lambda_{j}^{(3)}\left(T_{n}(f+g)\right)=\lambda_{j}\left(T_{n}(f)\right)+\lambda_{j}\left(T_{n}(g)\right)+h Q_{1}\left(\sigma_{j, n}\right)+h^{2} Q_{2}\left(\sigma_{j, n}\right) .
\end{aligned}
$$

Consider the error terms $\varepsilon_{j, n}^{(k)} \equiv \lambda_{j}\left(T_{n}(f+g)\right)-\lambda_{j}^{(k)}\left(T_{n}(f+g)\right)$, and the corresponding maximal absolute error $\varepsilon_{n}^{(k)}$.
We must have $\varepsilon_{j, n}^{(k)}=O\left(h^{k}\right)$ for $k=1,2,3$ uniformly in $j$, more specifically, the normalized errors $(n+1)^{k} \varepsilon_{j, n}^{(k)}$ for $k=1,2$ must be "close" to the terms $Q_{1}$ and $Q_{2}$, respectively.

Numerical verifications: $f+g$


The term $Q_{1}$ versus the normalized error $(n+1) \varepsilon_{j, n}^{(1)}$ for $n=128$.

Numerical verifications: $f+g$


The term $Q_{2}$ versus the normalized error $(n+1)^{2} \varepsilon_{j, n}^{(2)}$ for $n=128$.

Numerical verifications: $f+g$


The normalized error $(n+1)^{3} \varepsilon_{j, n}^{(3)}$ for $n=128$.

## Numerical verifications: $f+g$

| $n$ | 1024 | 2048 | 4096 | 8192 |
| :---: | :---: | :---: | :--- | :--- |
| $\varepsilon_{n}^{(1)}$ | $1.1753 \times 10^{-4}$ | $5.8918 \times 10^{-5}$ | $2.9498 \times 10^{-5}$ | $1.4759 \times 10^{-5}$ |
| $\hat{\varepsilon}_{n}^{(1)}$ | $1.2047 \times 10^{-1}$ | $1.2072 \times 10^{-1}$ | $1.2085 \times 10^{-1}$ | $1.2092 \times 10^{-1}$ |
| $\varepsilon_{n}^{(2)}$ | $6.1091 \times 10^{-7}$ | $1.5295 \times 10^{-7}$ | $3.8265 \times 10^{-8}$ | $9.5697 \times 10^{-9}$ |
| $\hat{\varepsilon}_{n}^{(2)}$ | $6.4184 \times 10^{-1}$ | $6.4214 \times 10^{-1}$ | $6.4229 \times 10^{-1}$ | $6.4237 \times 10^{-1}$ |
| $\varepsilon_{n}^{(3)}$ | $1.0885 \times 10^{-9}$ | $1.3634 \times 10^{-10}$ | $1.7059 \times 10^{-11}$ | $2.1231 \times 10^{-12}$ |
| $\hat{\varepsilon}_{n}^{(3)}$ | $1.1722 \times 10^{0}$ | $1.1729 \times 10^{0}$ | $1.1732 \times 10^{0}$ | $1.1736 \times 10^{0}$ |

## Numerical verifications: $f+g h$

We need to calculate the singular integral in $\psi$, which for this example can be simplified to

$$
\psi(s)=\frac{\sin (s)}{\pi} \int_{0}^{2 \pi} \frac{1}{f(\sigma)-f(s)} \mathrm{d} \sigma
$$

For the $k$ th term approximation we get this time

$$
\begin{aligned}
& \lambda_{j}^{(1)}=f\left(\sigma_{j, n}\right), \\
& \lambda_{j}^{(2)}=f\left(\sigma_{j, n}\right)+\Psi_{1}\left(\sigma_{j, n}\right) h, \\
& \lambda_{j}^{(3)}=f\left(\sigma_{j, n}\right)+\Psi_{1}\left(\sigma_{j, n}\right) h+\Psi_{2}\left(\sigma_{j, n}\right) h^{2} .
\end{aligned}
$$

We must have $\varepsilon_{j, n}^{(k)}=O\left(h^{k}\right)$ uniformly in $j=1, \ldots, n$, more specifically the normalized errors $(n+1)^{k} \varepsilon_{j, n}^{(k)}$ for $k=1,2$ are expected to be "close" to $\Psi_{1}$ and $\Psi_{2}$, respectively.

## Numerical verifications: $f+g h$



The term $\Psi_{1}$ versus the normalized error $(n+1) \varepsilon_{j, n}^{(1)}$ for $n=128$.

## Numerical verifications: $f+g h$



The term $\Psi_{2}$ versus the normalized error $(n+1)^{2} \varepsilon_{j, n}^{(2)}$ for $n=128$.

## Numerical verifications: $f+g h$



The normalized error $(n+1)^{3} \varepsilon_{j, n}^{(3)}$ for $n=128$.

## Numerical verifications: $f+g h$

| $n$ | 1024 | 2048 | 4096 | 8192 |
| :---: | :--- | :--- | :--- | :--- |
| $\varepsilon_{n}^{(1)}$ | $3.9024 \times 10^{-3}$ | $1.9522 \times 10^{-3}$ | $9.7632 \times 10^{-4}$ | $4.8822 \times 10^{-4}$ |
| $\hat{\varepsilon}_{n}^{(1)}$ | $4.0000 \times 10^{0}$ | $4.0000 \times 10^{0}$ | $4.0000 \times 10^{0}$ | $4.0000 \times 10^{0}$ |
| $\varepsilon_{n}^{(2)}$ | $1.2498 \times 10^{-6}$ | $3.1315 \times 10^{-7}$ | $7.8377 \times 10^{-8}$ | $1.9605 \times 10^{-8}$ |
| $\hat{\varepsilon}_{n}^{(2)}$ | $1.3130 \times 10^{0}$ | $1.3147 \times 10^{0}$ | $1.3156 \times 10^{0}$ | $1.3160 \times 10^{0}$ |
| $\varepsilon_{n}^{(3)}$ | $3.3384 \times 10^{-9}$ | $4.2375 \times 10^{-10}$ | $5.6220 \times 10^{-11}$ | $7.5973 \times 10^{-12}$ |
| $\hat{\varepsilon}_{n}^{(3)}$ | $3.5950 \times 10^{0}$ | $3.6453 \times 10^{0}$ | $3.8662 \times 10^{0}$ | $3.9850 \times 10^{0}$ |

## Numerical verifications: $f+g h^{h}$

We need to calculate the singular integral in $\varphi$, which for this example can be simplified to

$$
\varphi(s)=\frac{\sin (s)}{\pi} \int_{0}^{2 \pi} \frac{1}{f(\sigma)-f(s)+2 \cos (s)-2 \cos (\sigma)} \mathrm{d} \sigma .
$$

Taking into account that the logarithm is relatively small for the matrix sizes considered, we arrange the $k$ th term approximation in a different way this time

$$
\begin{aligned}
& \lambda_{j}^{(1)}=f\left(\sigma_{j, n}\right)+g\left(\sigma_{j, n}\right), \\
& \lambda_{j}^{(2)}=f\left(\sigma_{j, n}\right)+g\left(\sigma_{j, n}\right)+\Gamma_{1,1}\left(\sigma_{j, n}\right) h \log (h)+\Gamma_{1,0}\left(\sigma_{j, n}\right) h .
\end{aligned}
$$

## Numerical verifications: $f+g h^{h}$

We need to calculate the singular integral in $\varphi$, which for this example can be simplified to

$$
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& \lambda_{j}^{(2)}=f\left(\sigma_{j, n}\right)+g\left(\sigma_{j, n}\right)+\Gamma_{1,1}\left(\sigma_{j, n}\right) h \log (h)+\Gamma_{1,0}\left(\sigma_{j, n}\right) h .
\end{aligned}
$$

We have the bound $\varepsilon_{j, n}^{(k)}=O\left(h^{k}\left|\log ^{k}(h)\right|\right)$, more specifically the normalized errors $\frac{(n+1)^{k}}{\log ^{k}(n+1)} \varepsilon_{j, n}^{(k)}$ for $k=1,2$ are expected to be "close" to the functions $\Gamma_{1,1}+\Gamma_{1,0} \frac{1}{\log (h)}$ and $\Gamma_{2,2}+\Gamma_{2,1} \frac{1}{\log (h)}+\Gamma_{2,0} \frac{1}{\log ^{2}(h)}$, respectively.

Numerical verifications: $f+g h^{h}$


The term $\Gamma_{1}$ versus the normalized error $\frac{n+1}{\log (n+1)} \varepsilon_{j, n}^{(1)}$ for $n=128$.

## Numerical verifications: $f+g h^{h}$



The term $\Gamma_{2}$ versus the normalized error $\frac{(n+1)^{2}}{\log ^{2}(n+1)} \varepsilon_{j, n}^{(2)}$ for $n=128$.

Numerical verifications: $f+g h^{h}$


The normalized error $\frac{(n+1)^{3}}{\log ^{3}(n+1)} \varepsilon_{j, n}^{(3)}$ for $n=128$.

## Numerical verifications: $f+g h^{h}$

| $n$ | 1024 | 2048 | 4096 | 8192 |
| :---: | :--- | :--- | :--- | :--- |
| $\varepsilon_{n}^{(1)}$ | $2.6962 \times 10^{-2}$ | $1.4858 \times 10^{-2}$ | $8.1128 \times 10^{-3}$ | $4.3970 \times 10^{-3}$ |
| $\hat{\varepsilon}_{n}^{(1)}$ | $3.9865 \times 10^{0}$ | $3.9926 \times 10^{0}$ | $3.9959 \times 10^{0}$ | $3.9978 \times 10^{0}$ |
| $\varepsilon_{n}^{(2)}$ | $9.1280 \times 10^{-5}$ | $2.7663 \times 10^{-5}$ | $8.2384 \times 10^{-6}$ | $2.4184 \times 10^{-6}$ |
| $\hat{\varepsilon}_{n}^{(2)}$ | $1.9955 \times 10^{0}$ | $1.9975 \times 10^{0}$ | $1.9986 \times 10^{0}$ | $1.9993 \times 10^{0}$ |
| $\varepsilon_{n}^{(3)}$ | $2.0621 \times 10^{-7}$ | $3.4466 \times 10^{-8}$ | $5.6243 \times 10^{-9}$ | $9.0250 \times 10^{-10}$ |
| $\hat{\varepsilon}_{n}^{(3)}$ | $6.6654 \times 10^{-1}$ | $6.6877 \times 10^{-1}$ | $6.7206 \times 10^{-1}$ | $6.7835 \times 10^{-1}$ |

## A non-simple-loop symbol

For $k \in \mathbb{Z}_{+}$let $f_{k}(\theta) \equiv(2-2 \cos (\theta))^{k}$ and $\alpha_{k} \in \mathbb{R}$. Consider the symbol

$$
F_{n}(\theta) \equiv f_{2}(\theta)+\alpha_{1} f_{1}(\theta) h^{2}+\alpha_{0} f_{0}(\theta) h^{4} .
$$

The related Toeplitz matrices $T_{n}\left(F_{n}\right)$ appear when discretizing differential equations with Finite Differences.

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$$

The related Toeplitz matrices $T_{n}\left(F_{n}\right)$ appear when discretizing differential equations with Finite Differences.

The functions $f_{k}$ with $k \neq 2$, do not belong to $\mathrm{SL}^{\alpha}$ for any $\alpha$, thus $F_{n}$ do not fully satisfy our hypothesis and we can not apply our theoretical results, there is numerical evidence suggesting that, nevertheless we can expect an eigenvalue expansion of the form

$$
\lambda_{j}\left(T_{n}\left(F_{n}\right)\right)=f_{2}\left(\sigma_{j, n}\right)+\sum_{\ell=1}^{3} c_{\ell}\left(\sigma_{j, n}\right) h^{\ell}+E_{\alpha, j, n}
$$

where $E_{\alpha, j, n}=O\left(h^{4}\right)$ means asymptotic expansion.

## A non-simple-loop symbol

$$
\hat{\lambda}_{j}\left(T_{n}\left(F_{n}\right)\right)=f_{2}\left(\sigma_{j, n}\right)+\sum_{\ell=1}^{3} c_{\ell}\left(\sigma_{j, n}\right) h^{\ell}
$$



The 10-base logarithm for $\left|\lambda_{j}\left(T_{n}\left(F_{n}\right)\right)-\hat{\lambda}_{j}\left(T_{n}\left(F_{n}\right)\right)\right|$ with $n=8192$.

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