Bounded Symmetric Domains: Biholomorphisms, Spaces and Representations

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- We will also consider D endowed with the Lebesgue measure $\mathrm{d}v(z)$ normalized so that $v(D) = 1$.
- The domain D is called a **bounded symmetric domain** if for every $z \in D$ there exists a biholomorphism $\varphi \in \mathrm{Aut}(D)$ such that $\varphi(z) = z$ and $\varphi(w) \neq w$ for all $w \in D \setminus \{z\}$.

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- BSDs are part of a larger family of special manifolds.
- Let M be a Riemannian manifold.
  - M is called a **Riemannian symmetric space** if for every $x \in M$ there exists an isometry $\varphi \in \mathrm{Iso}(M)$ such that $\varphi(x) = x$ and $\varphi(y) \neq y$ for every $y \neq x$ **in a neighborhood of** x.

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  - M is called a **Hermitian symmetric space** if it is both a complex manifold and a Riemannian symmetric space so that the Riemannian metric is a Kähler metric.

- A domain $D \subset \mathbb{C}^N$ is called **reducible** if there is a domain of the form $D_1 \times D_2$ where $D_j \subset \mathbb{C}^{N_j}$, $j = 1, 2$ and $N_1, N_2 \geq 1$, such that $D \simeq D_1 \times D_2$ ($\simeq$ means biholomorphically equivalent). Otherwise, the domain D is called **irreducible**.

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Theorem

If D is a BSD, then there exist $D_1, \ldots, D_k$ irreducible domains such that $D \simeq D_1 \times \cdots \times D_k$. Furthermore, each factor D_j is a BSD, and the decomposition is unique up to biholomorphisms and permutations of the factors.

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Proof.

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Corollary

To enumerate the BSDs it is enough to enumerate the irreducible BSDs.

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Theorem (Borel embedding theorem)

For every BSD $D \subset \mathbb{C}^N$ and for its compact dual M (a Hermitian symmetric space), there is a biholomorphic open embedding $D \subset \mathbb{C}^N \hookrightarrow M$.

The unit ball

- The unit ball $\mathbb{B}^n = \{z \in \mathbb{C}^n \mid |z| < 1\}$ has the Borel embedding $\varphi : \mathbb{B}^n \hookrightarrow \mathbb{CP}^n$ given by

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$$\iff \overline{w}' \cdot w' < |w_{n+1}|^2 \iff \text{ the line } \mathbb{C}w \text{ is negative definite for } \langle \cdot, \cdot \rangle_{n,1}$$
where $\langle \cdot, \cdot \rangle_{n,1}$ denotes the Hermitian form on $\mathbb{C}^{n+1}$ given by
$$\langle a, b \rangle_{n,1} = \sum_{j=1}^{n} \overline{a}_j b_j - \overline{a}_{n+1} b_{n+1}.$$

- On $\mathbb{C}^{n+m}$, let us consider the Hermitian form given by

$$\langle a, b\rangle_{n,m} = a^* I_{n,m} b,$$

where $I_{n,m} = \mathrm{diag}(I_n, -I_m)$.

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- Alternatively, let us denote with $M_{(n+m)\times m}(\mathbb{C})^*$ the set of rank m elements of $M_{(n+m)\times m}(\mathbb{C})$ and define the equivalence relation

$$W_1 \simeq W_2 \iff \exists\, A \in \mathrm{GL}(m, \mathbb{C}) \text{ such that } W_1 = W_2 A.$$

Then, the complex Grassmanniannan in question is given by $\mathrm{Gr}_{\mathbb{C}}(n+m, m) = M_{(n+m)\times m}(\mathbb{C})^* / \simeq$.

- There is a natural embedding $M_{n\times m}(\mathbb{C}) \hookrightarrow \mathrm{Gr}_{\mathbb{C}}(n+m,m)$ given by

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Cartan domains of type I

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- Problem: Find the elements of $\mathrm{Gr}_{\mathbb{C}}(n+m, m)$ that are negative definite with respect to $\langle\cdot,\cdot\rangle_{n,m}$.

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- Solution: For $Z \in M_{n\times m}(\mathbb{C})$ we have

$$\begin{bmatrix} Z \\ I_m \end{bmatrix} \text{ is negative definite}$$

$$\iff (Z^*, I_m) I_{n,m} \begin{pmatrix} Z \\ I_m \end{pmatrix} < 0 \iff (Z^*, -I_m) \begin{pmatrix} Z \\ I_m \end{pmatrix} < 0$$

$$\iff Z^* Z < I_m.$$

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- Note that $\mathbb{B}^n = D^I_{n\times 1}$.
- The **Borel embedding** in this case is given by

$$\begin{aligned} D^I_{n\times m} &\hookrightarrow \mathrm{Gr}_{\mathbb{C}}(n+m, m) \\ Z &\mapsto \begin{bmatrix} Z \\ I_m \end{bmatrix}, \end{aligned}$$

and proves that $D^I_{n\times m}$ is the set of m-**dimensional subspaces** of $\mathbb{C}^{n+m}$ that are **negative definite** for $\langle\cdot,\cdot\rangle_{n,m}$.

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and proves that $D^I_{n\times m}$ is the set of m-**dimensional subspaces** of $\mathbb{C}^{n+m}$ that are **negative definite** for $\langle\cdot,\cdot\rangle_{n,m}$.
- We can use this to compute the biholomorphism group $\mathrm{Aut}(D^I_{n\times m})$.

- The biholomorphism group of $\mathrm{Gr}_{\mathbb{C}}(n+m, m)$ is given by linear transformations. More precisely, the action of $\mathrm{SL}(n+m, \mathbb{C})$ given by

$$M \cdot [W] = [MW],$$

where $M \in \mathrm{SL}(n+m, \mathbb{C})$ and $[W] \in \mathrm{Gr}_{\mathbb{C}}(n+m, m)$, realizes the biholomorphisms of $\mathrm{Gr}_{\mathbb{C}}(n+m, m)$.

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Proposition

For the realization of $D^I_{n\times m}$ as an open subset of $\mathrm{Gr}_{\mathbb{C}}(n+m,m)$, the group $\mathrm{Aut}(D^I_{n\times m})$ is given by the action of $\mathrm{SU}(n,m)$

$$M \cdot [W] = [MW]$$

where $M \in \mathrm{SU}(n,m)$ and $[W] \in D^I_{n\times m}$.

Corollary

For $D^I_{n\times m} = \{Z \in M_{n\times m}(\mathbb{C}) \mid Z^*Z < I_m\}$, **the biholomorphism group** $\mathrm{Aut}(D^I_{n\times m})$ **is realized by the action of** $\mathrm{SU}(n, m)$ **given by**

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}$$

where A and D have sizes $n \times n$ and $m \times m$, respectively.

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Proof.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{bmatrix} Z \\ I_m \end{bmatrix} = \begin{bmatrix} AZ+B \\ CZ+D \end{bmatrix} = \begin{bmatrix} (AZ+B)(CZ+D)^{-1} \\ I_m \end{bmatrix} \mapsto (AZ+B)(CZ+D)^{-1}.$$

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Cartan domains of type II and III

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  - $\mathrm{SO}(2n, \mathbb{C}) = \{M \in \mathrm{SL}(2n, \mathbb{C})) \mid M^\top S_n M = S_n\}$.
- We consider the submanifolds of $\mathrm{Gr}_{\mathbb{C}}(2n, n)$ consisting of **isotropic subspaces** for either of the bilinear forms defined by J_n and S_n.

- We denote by $\mathrm{LGr}_{\mathbb{C}}(n)$ the subspace of $\mathrm{Gr}_{\mathbb{C}}(2n, n)$ consisting of the elements $[W]$ where the anti-symmetric bilinear form defined by J_n vanishes. This is equivalent to $W^\top J_n W = 0$.

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Proposition

The space $\mathrm{LGr}_{\mathbb{C}}(n)$ is a complex submanifold of $\mathrm{Gr}_{\mathbb{C}}(2n, n)$, whose group of biholomorphisms is realized by the action of $\mathrm{Sp}(n, \mathbb{C})$

$$M \cdot [W] = [MW],$$

where $M \in \mathrm{Sp}(n, \mathbb{C})$ and $[W] \in \mathrm{LGr}_{\mathbb{C}}(n)$.

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- Solution: For $Z \in M_{n\times n}(\mathbb{C})$

$$\begin{bmatrix} Z \\ I_n \end{bmatrix} \text{ is isotropic for } J_n$$

$$\iff (Z^\top, I_n) J_n \begin{pmatrix} Z \\ I_n \end{pmatrix} = 0 \iff (I_n, -Z^\top) \begin{pmatrix} Z \\ I_n \end{pmatrix} = 0$$

$$\iff Z = Z^\top.$$

- The **Cartan domain of type III** D_n^{III} is the set of n-**dimensional subspaces** of $\mathbb{C}^{2n}$ that are both **Lagrangian and negative definite** for the Hermitian form $\langle \cdot, \cdot \rangle_{n,n}$

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- As before, this allows us to compute $\mathrm{Aut}(D_n^{III})$.

Proposition

For $D_n^{III} = \{Z \in M_{n\times n}(\mathbb{C}) \mid Z = Z^\top,\ \overline{Z}Z < I_n\}$, **the biholomorphism group** $\mathrm{Aut}(D_n^{III})$ **is realized by the action of** $\mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{SU}(n,n)$ **given by**

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ+B)(CZ+D)^{-1},$$

where A, B, C, D *all have size* $n \times n$.

Proposition

For $D_n^{III} = \{Z \in M_{n\times n}(\mathbb{C}) \mid Z = Z^\top,\ \overline{Z}Z < I_n\}$, **the biholomorphism group** $\mathrm{Aut}(D_n^{III})$ **is realized by the action of** $\mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{SU}(n,n)$ **given by**

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- There is an isomorphism of Lie groups $\mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{SU}(n,n) \simeq \mathrm{Sp}(n,\mathbb{R}) = \{X \in \mathrm{GL}(2n,\mathbb{R}) \mid X^\top J_n X = J_n\}$.

- We denote by $\mathrm{OGr}_{\mathbb{C}}(n)$ the subspace of $\mathrm{Gr}_{\mathbb{C}}(2n, n)$ consisting of the elements $[W]$ where the symmetric bilinear form defined by S_n vanishes. This is equivalent to $W^\top S_n W = 0$.

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Proposition

The space $\mathrm{OGr}_{\mathbb{C}}(n)$ is a complex submanifold of $\mathrm{Gr}_{\mathbb{C}}(2n, n)$, whose group of biholomorphisms is realized by the action of $\mathrm{SO}(2n, \mathbb{C})$

$$M \cdot [W] = [MW],$$

where $M \in \mathrm{SO}(2n, \mathbb{C})$ and $[W] \in \mathrm{OGr}_{\mathbb{C}}(n)$.

- As before, for the natural embedding $M_{n\times n}(\mathbb{C}) \hookrightarrow \mathrm{Gr}_{\mathbb{C}}(2n, n)$ given by

$$Z \mapsto \begin{bmatrix} Z \\ I_n \end{bmatrix},$$

we have for every $Z \in M_{n\times n}(\mathbb{C})$

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- In particular, we have

$$D_n^{II} = \{Z \in M_{n\times n}(\mathbb{C}) \mid Z = -Z^\top,\ Z^*Z < I_n\},$$

and its Borel embedding is given as before.

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- There is an isomorphism of Lie groups $\mathrm{SO}(2n,\mathbb{C}) \cap \mathrm{SU}(n,n) \simeq \mathrm{SO}^*(2n)$, where the latter is a Lie group associated to a quaternionic structure.

Cartan domains of type IV

- The compact complex manifold for this type is given by

$$Q^n : \quad [z] \in \mathbb{CP}^{n+1} \text{ such that } \sum_{j=1}^{n} z_j^2 - 2z_{n+1}z_{n+2} = 0,$$

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$$D_n^{IV} = \{z \in \mathbb{C}^n \mid |z| < 1,\ 2|z|^2 < 1 + |z^\top z|^2\}.$$

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  - For the given realizations, D is a circular domain: $0 \in D$ and $tD = D$ for every $t \in \mathbb{T}$.
  - D is homogeneous: the action of $\mathrm{Aut}(D)$ is transitive.
  - The action of $\mathrm{Aut}(D)$ has been given as an action of a connected Lie group G. The subgroup K of G that fixes the origin acts linearly on D and yields a quotient $D = G/K$.

Cartan domains and Lie groups

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  - Type IV, D^{IV}_{n}: $G = \mathrm{SO}(n,2)$ and $K = \mathrm{SO}(n)\times\mathrm{SO}(2)\simeq\mathrm{SO}(n)\times\mathbb{T}$ acting by
$$(A,t)\cdot z = tAz.$$

Bergman spaces

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- One usually assumes that D bounded and take the normalized Lebesgue measure $\mathrm{d}v(z)$ so that $v(D) = 1$.
- The boundedness of D ensures that $\mathcal{P}(\mathbb{C}^N) \subset \mathcal{A}^2(D)$. And the condition $v(D) = 1$ simplifies the expression of the Bergman kernel.

Proposition

Let $D \subset \mathbb{C}^N$ be a domain. Then, for every compact subset $K \subset D$ there exists a constant $C_K > 0$ such that $\|f\|_{\infty,K} \leq C_k \|f\|_2$, for every $f \in \mathcal{A}^2(D)$. In particular, the following hold

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Proof.

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Corollary

The same conclusions hold for "weighted" Bergman spaces of the form $\mathcal{A}^2_w(D) = L^2(D, w(z)\,\mathrm{d}z) \cap \mathrm{Hol}(D)$, where $w : D \to (0, +\infty)$ is any continuous function.

- Let $D \subset \mathbb{C}^N$ be a given domain. For every $z \in D$, there exists $K_z \in \mathcal{A}^2(D)$ such that

$$f(z) = \int_D f(w)\overline{K_z}(w)\,\mathrm{d}v(z),$$

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Corollary

The orthogonal projection $B_D : L^2(D) \to \mathcal{A}^2(D)$ is given by
$$B_D(f)(z) = \int_D f(w)K(z,w)\,\mathrm{d}v(w),$$
for every $z \in D$. It is called the ***Bergman projection***.

Proposition

The biholomorphism group $\mathrm{Aut}(D)$ *has a unitary representation* π *on* $\mathcal{A}^2(D)$ *given by*

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For any domain D *with Bergman kernel* K *and* $\varphi \in \mathrm{Aut}(D)$ *we have*

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- Both results are obtained applying the change of variable theorem.

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Proposition

For any irreducible BSD D there exist two invariants, the genus p and the Jordan triple determinant $\Delta : D\times D\to\mathbb{C}$, such that

$$K(z,w)=\Delta(z,w)^{-p},$$

for every $z,w\in D$.

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- There are two exceptional irreducible BSD whose dimensions are 16 and 27, with genus 12 and 26, respectively.

- Let D be an irreducible BSD with Bergman kernel $K(z,w) = \Delta(z,w)^{-p}$. The most natural weight to consider is

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Proposition

For every $\lambda > p-1$, we have

$$\int_D \Delta(z,z)^{\lambda-p}\, \mathrm{d}z < \infty.$$

In particular, for every $\lambda > p-1$, there is a constant $c_\lambda > 0$ such that $\mathrm{d}v_\lambda(z) = c_\lambda \Delta(z,z)^{\lambda-p}\, \mathrm{d}z$ is a probability measure.

- Let D be an irreducible BSD with Bergman kernel $K(z,w) = \Delta(z,w)^{-p}$. The most natural weight to consider is

$$z \mapsto \Delta(z,z)^{\lambda - p} = K(z,z)^{1-\frac{\lambda}{p}}.$$

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Proof.

The value of the integral can be computed in terms of Gamma functions. □

- Let D be an irreducible BSD with genus p and Jordan triple determinant Δ. For every $\lambda > p-1$, the **weighted Bergman space with weight** λ is denoted by $\mathcal{A}^2_\lambda(D)$ and is given by

$$\mathcal{A}^2_\lambda(D) = L^2(D, v_\lambda) \cap \mathrm{Hol}(D)$$

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    - $\mathcal{A}^2_\lambda(D)$ is a closed subspace of $L^2(D, v_\lambda)$.
    - The evaluation functionals $f \mapsto f(z)$ are continuous on $\mathcal{A}^2_\lambda(D)$.
    - The orthogonal **Bergman projection** $B_\lambda : L^2(D, v_\lambda) \to \mathcal{A}^2_\lambda(D)$ is realized by a "weighted" Bergman kernel $K_\lambda : D \times D \to \mathbb{C}$.

Theorem

Let D be an irreducible BSD with genus p and (weightless) Bergman kernel $K(z,w) = \Delta(z,w)^{-p}$. Then, for every $\lambda > p-1$ the weighted Bergman kernel of $\mathcal{A}^2_\lambda(D)$ is given by

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- It is now an easy exercise to write down the weighted Bergman kernels for all classical Cartan domains.
- Since the BSDs are simply connected, it is easy to find branches of the λ-powers with the required analyticity.

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- For the unit ball and the Cartan domains of type II and III, we have $K = \mathrm{U}(n)$. Its universal covering map is given by

$$\begin{aligned} \mathbb{R} \times \mathrm{SU}(n) &\to \mathrm{U}(n) \\ (x, A) &\mapsto e^{ix} A. \end{aligned}$$

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- For every irreducible BSD $\widetilde{K} = \mathbb{R} \times L$ where L is a simply connected compact semisimple Lie group. The factor $\mathbb{R}$ covers a subgroup $\mathbb{T} \hookrightarrow K$.

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- For every $\lambda > p-1$, there is a unitary representation $\pi_\lambda : \widetilde{G} \times \mathcal{A}^2_\lambda(D) \to \mathcal{A}^2_\lambda(D)$ given by

$$\pi_\lambda(\varphi)(f) = J_{\mathbb{C}}(\varphi^{-1})^{\frac{\lambda}{p}}(f \circ \varphi^{-1}).$$

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Theorem

If $D = G/K$ is an irreducible BSD with genus p, then for every $\lambda > p - 1$ the unitary representation π_λ is irreducible: the spaces $\mathcal{A}^2_\lambda(D)$ and 0 are the only closed subspaces invariant under $\pi_\lambda(\varphi)$ for every $\varphi \in \widetilde{G}$.

- Let $H \subset \widetilde{G}$ be a closed subgroup. Then, we will denote by $\pi_\lambda|_H$ the restriction of π_λ to H, which yields a unitary representation $H \times \mathcal{A}^2_\lambda(D) \to \mathcal{A}^2_\lambda(D)$.

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- Problem: for an arbitrary closed subgroup $H \subset \widetilde{G}$ decompose $\mathcal{A}^2_\lambda(D)$ as a direct integral of irreducible unitary representations of H.
- Solution: depending on the group H this problem could be "unsolvable", with unknown solution, hard to solve or very well known.

The representation of K

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- If $\widetilde{K} \to K$ is the universal covering group, then for every $\varphi \in \widetilde{K}$ there exists $A \in K \subset \mathrm{GL}(N, \mathbb{C})$ such that $\varphi \mapsto A$ and so

$$J_{\mathbb{C}}(\varphi, z) = \det(A) \in \mathbb{T}$$

for every $z \in D$.

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Corollary

For every $\lambda > p - 1$, the representations of $\widetilde{K}$ and K on $\mathcal{A}^2_\lambda(D)$ given, respectively, by

$$(\varphi, f) \mapsto J_{\mathbb{C}}(f \circ \varphi^{-1})(f \circ \varphi^{-1}), \quad \text{and} \quad (A, f) \mapsto f \circ A^{-1},$$

have the same representation theoretic features. In particular, their decomposition into Hilbert directs sums of irreducible subspaces are the same.

The representation of K

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- Recall that $\mathcal{P}(\mathbb{C}^N) \subset \mathcal{A}^2_\lambda(D)$ is dense for every $\lambda > p-1$. Furthermore, we have
$$\pi_\lambda(A)(\mathcal{P}(\mathbb{C}^N)) = \mathcal{P}(\mathbb{C}^N),$$
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Corollary

The Hilbert direct sum decomposition into irreducible subspaces for the unitary representation $\pi_\lambda|_K$ on $\mathcal{A}^2_\lambda(D)$ is given by the decomposition into irreducible subspaces for the representation
$$\pi_\lambda|_K : K \times \mathcal{P}(\mathbb{C}^N) \to \mathcal{P}(\mathbb{C}^N).$$

- Recall that $\mathbb{T} \subset K$ acting linearly, so that

$$\pi_\lambda(t)(f)(z) = f(t^{-1}z)$$

for every $t \in \mathbb{T}$, $f \in \mathcal{A}^2_\lambda(D)$ and $z \in D$.

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- Let us denote by $\mathcal{P}^m(\mathbb{C}^N)$ the space of homogeneous polynomials of degree m in $\mathbb{C}^N$. It follows that the direct sums

$$\mathcal{P}(\mathbb{C}^N) = \bigoplus_{m=0}^{\infty} \mathcal{P}^m(\mathbb{C}^N), \qquad \text{algebraic direct sum,}$$

$$\mathcal{A}^2_\lambda(D) = \bigoplus_{m=0}^{\infty} \mathcal{P}^m(\mathbb{C}^N), \qquad \text{Hilbert direct sum,}$$

are both invariant under the representation $\pi_\lambda|_K$.

Theorem

For $D = G/K$ and for every $\lambda > p-1$, there is a $\pi_\lambda|_K$-invariant Hilbert direct sum

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- *If $V_j \subset \mathcal{P}^{m_j}(\mathbb{C}^N)$, $j = 1, 2$, are irreducible K-submodules and $m_1 \neq m_2$, then $V_1 \simeq V_2$ as K-modules.*

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- Conclusion: To obtain the decomposition of $\mathcal{A}^2_\lambda(D)$ into irreducible K-submodules, it is enough to study the representation $K \times \mathcal{P}^m(\mathbb{C}^N) \to \mathcal{P}^m(\mathbb{C}^N)$, for every $m \in \mathbb{N}$.

The representation of K

- There are very general Lie theoretic statements that describe the representation of K on the spaces $\mathcal{P}^m(\mathbb{C}^N)$. Nevertheless, for classical Cartan domains such statements can be made very explicit.

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The representation of K

- There are very general Lie theoretic statements that describe the representation of K on the spaces $\mathcal{P}^m(\mathbb{C}^N)$. Nevertheless, for classical Cartan domains such statements can be made very explicit.
- The first key point is to understand the representation $K \to \mathrm{GL}(N, \mathbb{C})$.
- Next, one uses the so called Invariant Theory.

The representation of K

- We can describe some general features of the representation $K \to \mathrm{GL}(N, \mathbb{C})$ for the classical Cartan domains.

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  - Cartan domains of type I, $D^I_{n\times m}$. In this case, we have $K = \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(m))$,

$$\mathbb{C}^N = M_{n\times m}(\mathbb{C}) \simeq L(\mathbb{C}^m, \mathbb{C}^n),$$

and the representation of K is given by

$$(A, B) \cdot Z = AZB^{-1} \simeq A \circ T_Z \circ B^{-1}$$

where T_Z is the linear transformation with matrix representation Z. This is the usual action obtained from changes of (unitary) coordinates.

The representation of K

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  - Cartan domains of type III, D_n^{II}. In this case, we have $K = \mathrm{U}(n)$,

$$\mathbb{C}^N = SM(n, \mathbb{C}) \simeq \mathrm{Sym}(\mathbb{C}^n),$$

and the representation of K is given by

$$A \cdot Z = AZA^\top \simeq B_Z(A(\cdot), A(\cdot)),$$

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  - Cartan domains of type II, D_n^{II}. Replace "symmetric" by "anti-symmetric" everywhere.

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The representation of K

- For the Cartan domains of type IV we have $K = \mathrm{SO}(n) \times \mathrm{SO}(2)$.
- This requires to study the natural representation $\mathrm{SO}(n) \hookrightarrow \mathrm{GL}(n, \mathbb{C})$.
- This is a classical problem, and the representations $\mathrm{SO}(n) \to \mathrm{GL}(\mathcal{P}^m(\mathbb{C}^n))$ can be studied using harmonic polynomials.

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