Pseudodistance associated to the reproducing kernel

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International Workshop on Operator Theory on Function Spaces
joint results with Erick Lee Guzmán and Egor Maximenko

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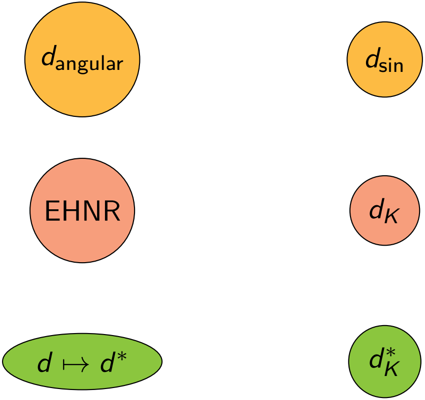
Objectives

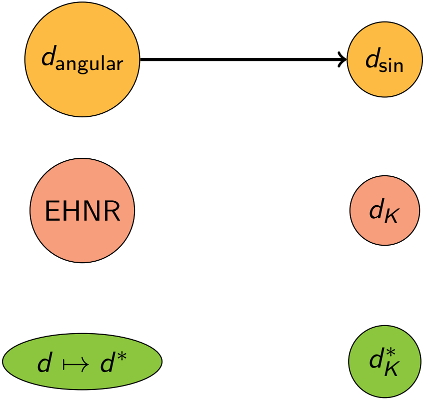
- Given a RKHS, to define

$$d_K(x, y) = \sqrt{1 - \frac{|K_x(y)|^2}{K_x(x)K_y(y)}}.$$

- To define intrinsic pseudodistance d_K^*.
- To compute d_K^* for several examples.

1. d_{sin}
2. Pseudodistance associated to a RK
3. Intrinsic pseudodistance
4. Example: Fock space
5. Riemannian metric


d_{angular}
$d_{\sin}$
EHNR
d_K
$d \mapsto d^*$
d_K^*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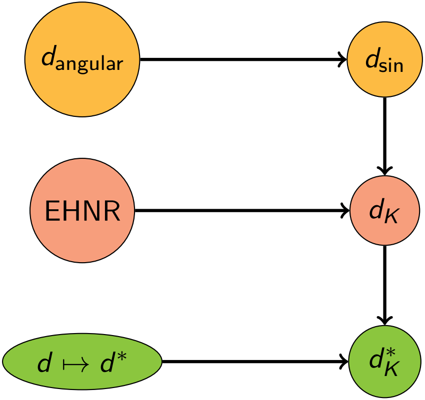
$$\begin{tikzcd}
d_{\text{angular}} \arrow[r] & d_{\sin} \arrow[d] \\
\text{EHNR} \arrow[r] & d_K \\
d \mapsto d^* & d_K^*
\end{tikzcd}$$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Content

Angular pseudodistnace

Definition

Let H be a Hilbert space. For $a, b \in H \setminus \{0\}$

$$d_{\text{angular}}(a, b) := \arccos \frac{|\langle a, b \rangle|}{\|a\| \|b\|}.$$

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It follows from the fact:

$$\det \mathrm{Gram}(a, b, c) = \begin{vmatrix} \langle a, a\rangle & \langle a, b\rangle & \langle a, c\rangle \\ \langle b, a\rangle & \langle b, b\rangle & \langle b, c\rangle \\ \langle c, a\rangle & \langle c, b\rangle & \langle c, c\rangle \end{vmatrix} \geq 0.$$

Sine pseudodistance

Definition

Let H be a Hilbert space. For $a, b \in H \setminus \{0\}$

$$d_{\sin}(a, b) := \sin(d_{\text{angular}}(a, b)) = \sqrt{1 - \frac{|\langle a, b\rangle|^2}{\|a\|^2\|b\|^2}}.$$

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Agler, McCarthy. Pick interpolation and Hilbert function spaces, 2002.

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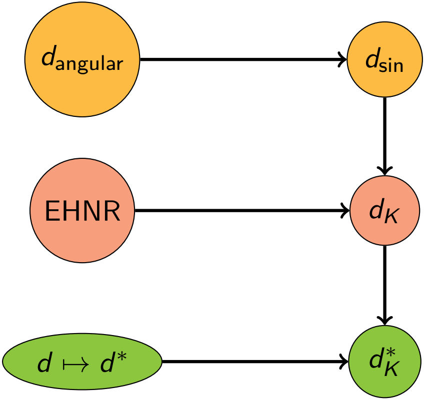
For each cada a, b in $H \setminus \{0\}$,

$$d_{\sin}(a, b) = \|P_a - P_b\|.$$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Reproducing kernel Hilbert space

Definition

Let X be a nonempty set and H a Hilbert space, $H \leq \mathbb{C}^X$.
A family $(K_x)_{x \in X}$ in H is called **reproducing kernel** if

$$\forall x \in X, \quad \forall f \in H, \quad \langle f, K_x \rangle = f(x).$$

Some properties of RKHS

Let H be a RKHS.

$$\|K_x\|^2 = \langle K_x, K_x \rangle = K_x(x).$$

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$$K_y(x) = \sum_{n\in\mathbb{N}} \overline{\varphi_n(y)}\varphi_n(x).$$

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Proposition

$\mathrm{span}(\{K_x : x \in X\})$ is dense in H.

Pseudodistance associated to a RKHS

Definition

Let $H \leq \mathbb{C}^X$ be a RKHS.
For $x, y \in X$, the pseudodistance d_K is defined on X by

$$d_K(x,y) \coloneqq d_{\sin}(K_x, K_y),$$

i.e.,

$$d_K(x,y) = \sqrt{1 - \frac{|\langle K_x, K_y\rangle|^2}{\|K_x\|^2\|K_y\|^2}} = \sqrt{1 - \frac{|K_x(y)|^2}{K_x(x)K_y(y)}}.$$

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An equivalent expression:

$$d_K(x, y) = \|P_{K_x} - P_{K_y}\|.$$

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Arcozzi, Rochberg, Sawyer, Wick.
Distance functions for reproducing kernel Hilbert spaces, 2011.

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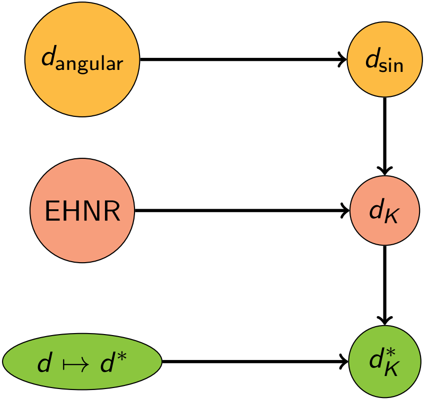
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Length of paths with respect to a pseudodistance

Let (X, d) be pseudometric space and $x, y \in X$.

$$\Gamma(x,y) := \Big\{\gamma \in C([0,1], X) : \quad \gamma(0) = x, \quad \gamma(1) = y\Big\}.$$

Given $x, y \in X$ and $\gamma \in \Gamma(x, y)$, the length of γ with respect to d is defined by

$$L(d,\gamma) := \sup_{0=t_0\le t_1\le\cdots\le t_n=1} \sum_{k=1}^{n} d(\gamma(t_k), \gamma(t_{k-1})).$$

Intrinsic pseudodistance induced by d

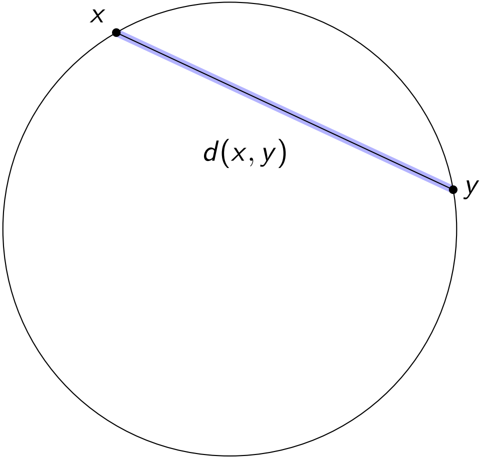
Definition

$d^* : X \times X \to \mathbb{R}$ is defined by

$$d^*(x, y) := \inf_{\gamma \in \Gamma(x,y)} L(d, \gamma).$$

Note $d(x, y) \leq d^*(x, y)$.

Example: Intrinsic distance in the unit circle



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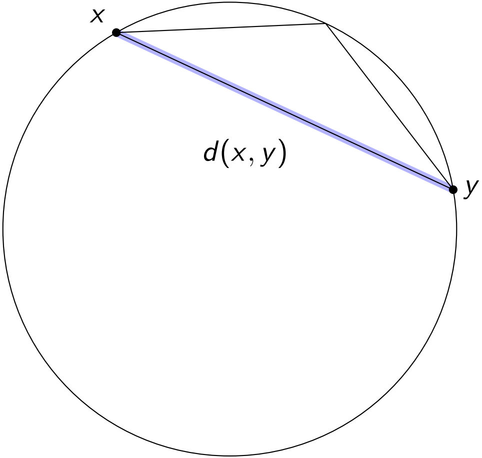

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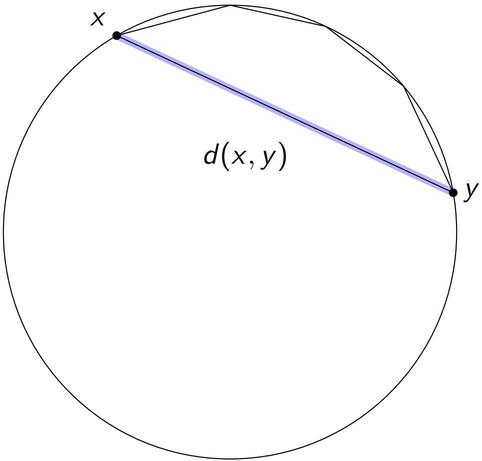

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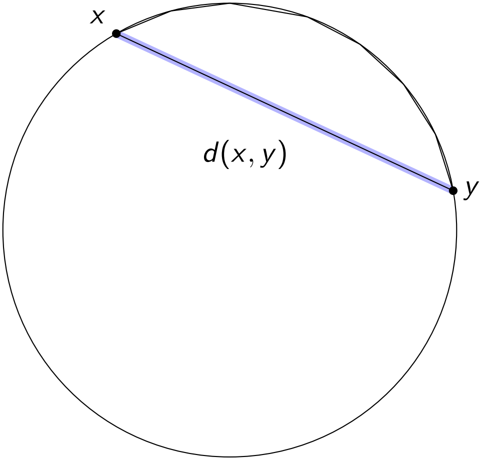

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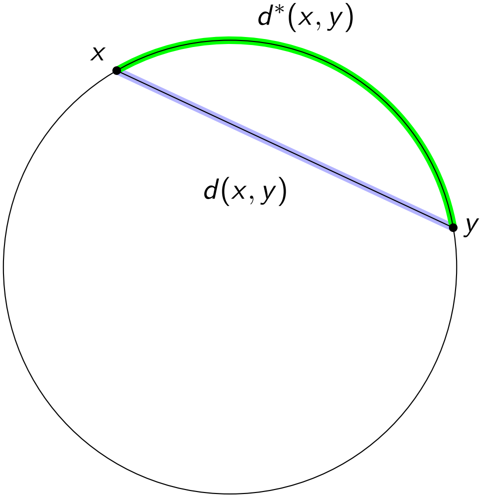

Locally similar pseudodistances

Definition

Let X be a nonempty set and d_1, d_2 be pseudodistances on X. We say d_1, d_2 are locally similar if $\exists\ C > 0$ such that $\forall\, x \in X$ the following limits relations hold:

$$\lim_{\substack{d_1(y,x)\to 0\\ d_1(z,x)\to 0\\ d_1(y,z)>0}} \frac{d_2(y,z)}{d_1(y,z)} = C, \qquad \lim_{\substack{d_2(y,x)\to 0\\ d_2(z,x)\to 0\\ d_2(y,z)>0}} \frac{d_1(y,z)}{d_2(y,z)} = \frac{1}{C}.$$

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$$\forall x \in X\ \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall y, z \in B_{d_1}(x,\delta) \quad \left| \frac{d_2(y,z)}{d_1(y,z)} - C \right| < \varepsilon,$$

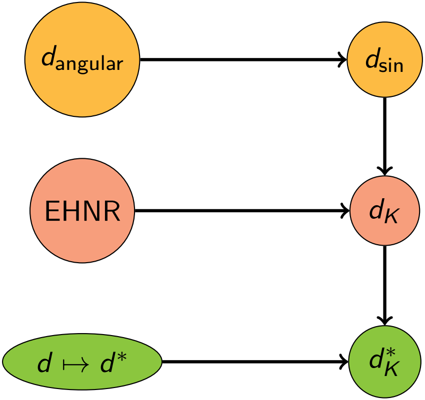
$$\forall x \in X,\ \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall y, z \in B_{d_2}(x,\delta) \quad \left| \frac{d_1(y,z)}{d_2(y,z)} - \frac{1}{C} \right| < \varepsilon.$$

Intrinsic pseudodistances induced by localli similar pseudodistances

It is easy to see that if d_1, d_2 are locally similar pseudodistances, they induce the same topology.

Theorem

Let X be a nonempty set and d_1, d_2 locally similar pseudodistances on X. Then there is $C > 0$ such that $d_1^* = C\, d_2^*$.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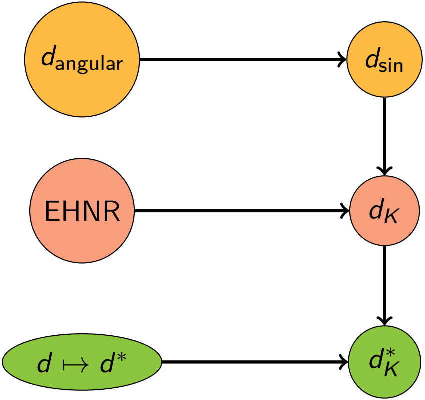

$d_K^* :=$ the intrinsic pseudodistance associated to d_K.

Content

Bargmann–Segal–Fock space

Consider the space

$$\mathcal{F}(\mathbb{C}) := \left\{ f \in \mathrm{Hol}(\mathbb{C}) : \quad \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} \, d\mu(z) < +\infty \right\}.$$

The reproducing kernel of $\mathcal{F}(\mathbb{C})$ is

$$K_z(w) = e^{\bar{z}w}.$$

Reproducing kernel and d_K

$$
\begin{aligned}
d_K(z,w) &= \sqrt{1 - \frac{|K_z(w)|^2}{K_z(z)K_w(w)}} \\
&= \sqrt{1 - \frac{|e^{\bar{z}w}|^2}{e^{|z|^2}e^{|w|^2}}} \\
&= \sqrt{1 - \exp(2\,\mathrm{Re}\{|\overline{w}z|\} - |z|^2 - |w|^2)} \\
&= \sqrt{1 - \exp(-|z-w|^2)}
\end{aligned}
$$

An approximation for d_K in $\mathcal{F}(\mathbb{C})$

We use the following approximation

$$1 - e^{-t} = 1 - \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} = t + O(t^2)$$

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Therefore

$$d_K(z, w) = \sqrt{1 - \exp(-|z - w|^2)} = |z - w| + O(|z - w|^3).$$

Proposition

d_K and the euclidean distance are locally similar.

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$$\lim_{\substack{d_K(y,x)\to 0\\ d_K(z,x)\to 0\\ y\neq z}} \frac{d(y,z)}{d_K(y,z)} = \lim_{\substack{d_K(y,x)\to 0\\ d_K(z,x)\to 0\\ y\neq z}} \frac{|y-z|}{|y-z| + O(|y-z|^3)} = 1$$

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$$\lim_{\substack{d(y,x)\to 0\\ d(z,x)\to 0\\ y\neq z}} \frac{d_K(y,z)}{d(y,z)} = \lim_{\substack{d(y,x)\to 0\\ d(z,x)\to 0\\ y\neq z}} \frac{|y-z| + O(|y-z|^3)}{|y-z|} = 1$$

$\square$

For this case, $C = 1$ y $d_K^* = d = d^*$.

More examples

- Bergman space over the unit disk:

$$d_K^* = \text{ hyperbolic distance.}$$

- Fock space over the complex plane:

$$d_K^* = \text{ euclidean distance.}$$

- Bandlimited functions over $\mathbb{R}$ with band $[-\pi, \pi]$:

$$d_K^* = \text{euclidean distance.}$$

- Finite-dimensional Hilbert spaces:

$$d_K^* = \text{discrete distance.}$$

- Sobolev space: $d_K(a, b) = \infty$ for $a \neq b$.

Content

Local approximation of d_K in the smooth case

Suppose $X \subseteq \mathbb{R}$ is an open set and $H \leq \mathbb{C}^X$ is a RKHS.

Suppose that for each $x_0 \in X$ there is an open neighborhood V of x_0 and a function $g \in C^2(V^2)$ such that for each $x, y \in V$

$$K_x(y) = \exp(g(y, x)).$$

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Proposition

With the previous suppositions, let $x_0 \in X$

$$d_K(x, y) = \sqrt{D_{1,2}g(x_0, x_0)}\,|x - y| + o(|x - y|),$$

when $x, y \to x_0$.

Proof: Because $g \in C^2(V^2)$, around x_0 for each $x, y \in V$

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$$\begin{aligned} g(x,y) =& g(x_0,x_0) + A_1(x-x_0) + A_2(y-x_0) \\ & + A_{11}(x-x_0)^2 + A_{12}(x-x_0)(y-x_0) + A_{22}(w-x_0)^2 \\ & + R(z,w). \end{aligned}$$

Here

$$A_j = D_j g(x_0,x_0), \quad j \in \{1,2\},$$

$$A_{j,k} = D_j D_k g(x_0,x_0), \quad j,k \in \{1,2\}.$$

$$
\begin{aligned}
K_x(y) &= \exp(g(y,x)) \\
&= \exp\Big(g(x_0,x_0) + A_1(z-x_0) + A_2(w-x_0) \\
&\quad + A_{11}(x-x_0)^2 + A_{12}(x-x_0)(y-z_0) + A_{22}(y-x_0)^2 \\
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\end{aligned}
$$

After simplifying the quotient

$$
\frac{K_x(y)K_y(x)}{K_x(x)K_y(y)} = \exp(A_{12}(y-x)^2 + r(x,y)).
$$

Therefore,

$$d_K(x,y) = \sqrt{1 - \exp(A_{12}((x-y)^2 + r(x,y)))}.$$

Writing exp as power series

$$d_K(x,y) = \sqrt{A_{12}(x-y)^2 + r'(x,y)} = |x-y|\sqrt{A_{12}} + o(|x-y|).$$

$$d_K(x,y) = |x-y|\sqrt{A_{12}} + o(|x-y|).$$

Corollary

d_K^* coincides with the geodesic distance associated to the Riemannian metric

$$\sqrt{D_{1,2}g(x,x)}\,\mathrm{d}\,x.$$