Weighted Bergman Spaces associated with the hyperbolic group

International Workshop on Operator Theory on Function Spaces

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Contenido

- Π denotes the upper half-plane in $\mathbb{C}$ with the usual Lebesgue area measure $d\mu(z) = dxdy$, $z = x + iy$.

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- We consider the following normalized invariant measure on the upper half-plane Π,

$$d\nu(z) = \frac{1}{\pi}\frac{dxdy}{(2\ \mathrm{Im} z)^2}.$$

- For each $\lambda \in (-1, \infty)$, the weighted Bergman space $\mathcal{A}^2_\lambda(\Pi)$ on the upper half-plane is the space of analytic functions in $L_2(\Pi, d\nu_\lambda)$, where

$$d\nu_\lambda(z) = (\lambda + 1)(2\mathrm{Im}(z))^{\lambda+2} d\nu(z) = c_\lambda (Im(z))^\lambda dxdy,$$

the normalizing constant is given by

$$c_\lambda = 2^\lambda \frac{\lambda + 1}{\pi},$$

And

$$||f||_\lambda = \left(\int_\Pi |f(z)|^2 d\nu_\lambda(z) \right)^{1/2}.$$

The weighted Bergman projection $B_{\Pi,\lambda} : L_2(\Pi, d\nu_\lambda) \longrightarrow \mathcal{A}^2_\lambda(\Pi)$ has the form

$$(B_{\Pi,\lambda} f)(z) = (\lambda + 1) \int_\Pi f(\zeta) \left(\frac{\zeta - \overline{\zeta}}{z - \overline{\zeta}} \right)^{\lambda+2} d\nu(\zeta).$$

Weighted poly-Bergman space

Let $\mathcal{A}^2_{\lambda,n}(\Pi)$ be the weighted poly-Bergman space that consists of all functions in $L_2(\Pi, d\nu)$ satisfying the equation

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The weighted true-poly-Bergman spaces are:

$$\mathcal{A}^2_{\lambda,(n)}(\Pi) = \mathcal{A}^2_{\lambda,n}(\Pi) \ominus \mathcal{A}^2_{\lambda,n-1}(\Pi), \quad n = 1, 2, \dots \tag{1}$$

where $\mathcal{A}^2_{\lambda,0}(\Pi) = \{0\}$.
$\mathcal{A}^2_{\lambda}(\Pi) = \mathcal{A}^2_{\lambda,(1)}(\Pi)$ is the usual Bergman space.

Contenido

Loiza and Ramírez-Ortega studied Poly-Bergman spaces on the upper half plane

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They found that $A_n^2 := \hat{U}(\mathcal{A}_n^2(\Pi)) \subset L_2(\mathbb{R}, dx) \otimes L_2([0,\pi], d\theta)$ consists of functions $f(x,\theta)$ satisfying

$$\left(ix - 1 - 2[n-1] + i\frac{\partial}{\partial\theta}\right) \cdots \left(ix - 1 + i\frac{\partial}{\partial\theta}\right) f = 0.$$

$$A^2_{(n)} = A^2_n \ominus A^2_{n-1} = \{f(x)p_{n-1}(x,\theta)\psi(x)e^{-x\theta-\theta i} \mid f(x) \in L_2(\mathbb{R}, dx)\}, \quad n = 1, 2, \ldots$$

where $p_n(x,\theta)$ is a polynomial with respect to $z = e^{-2\theta i}$

$$p_n(x,\theta) = \sum_{k=0}^{n} (-1)^k b_{nk}(x) e^{-2k\theta i}, n = 0, 1, 2, \ldots,$$

with $b_{00} = 1$,

$$b_{nn}(x) = \frac{1}{n!}\sqrt{(x^2+1)(x^2+2^2)\cdots(x^2+n^2)},$$

$$b_{nk}(x) = \binom{n}{k} b_{nn}(x) \prod_{j=0}^{n-1} \frac{x - ji + ki}{x - ji + ni}, \quad \psi(x) = \sqrt{\frac{2x}{1 - e^{-2\pi x}}}, \quad y \quad \psi(0) = \frac{1}{\sqrt{\pi}}.$$

They found:

- An orthonormal set $\{p_n(x,\theta) \mid n = 0,1,2,\ldots\}$ on the space $L_2\left([0,\pi], (\psi(x))^2 e^{-2x\theta} d\theta\right)$ for each $x \in \mathbb{R}$.

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- $A^2_{(n)}$ is the image of the pure poly-Bergman space $\mathcal{A}^2_{(n)}(\Pi)$ under the operator $\hat{U}$.
- $P_{(n)}$ is the orthogonal projection of $L_2(\mathbb{R}, dx) \otimes L_2([0,\pi], d\theta)$ onto $A^2_{(n)}$.

$$(P_{(n)}f)(x,\theta) = (\psi(x))^2 p_{n-1}(x,\theta) e^{-x\theta-\theta i} \int_0^\pi f(x,\varphi)\overline{p_{n-1}(x,\varphi)} e^{-x\varphi+\varphi i} d\varphi.$$

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- $P_n := \sum_{k=1}^n P_{(k)}$ is the orthogonal projection of $L_2(\mathbb{R}, dx) \otimes L_2([0,\pi], d\theta)$ onto A^2_n.

Contenido

Kähler form for the upper half-plane

$$(\omega_{\Pi})_z = \frac{i dz \wedge d\overline{z}}{2(\mathrm{Im}(z))^2}$$

where $z \in \Pi$.

Maximak Abelian subgroups for Π

Quasi-elliptic action: The $\mathbb{T}$-action on $\mathbb{D}$ given by

$$t \cdot z = tz.$$

Parabolic action: The $\mathbb{R}$-action on Π given by

$$x \cdot z = z + x.$$

Hyperbolic action: The $\mathbb{R}_+$-action on Π given by

$$r \cdot z = rz.$$

Let G be a connected Lie group acting smoothly on Π and preserving its Kähler form $(\omega_\Pi)_z$, with Lie algebra $\mathfrak{g}$. For every $X \in \mathbb{R} \cong \mathfrak{g}$, the G-action induces a smooth vector field given by

$$X_z^\sharp = \frac{d}{ds}\,|_{s=0}\, \exp(sX) \cdot z$$

for every z in the upper-half-plane, where $\cdot$ denotes the G-action on Π.

Definition

Let G be a maximal Abelian subgroup of biholomorphisms of Π. A moment map for the G-action is a smooth G-invariant map $\mu = \mu^G : \Pi \to \mathbb{R}$ such that

$$d\mu_X = \omega(X^\sharp, \cdot)$$

for every $X \in \mathbb{R}$, where ω is the Kähler form of Π and $\mu_X : \Pi \to \mathbb{R}$ is the smooth function given by

$$\mu_X(z) = \langle \mu(z), X \rangle,$$

for every $z \in \Pi$.

On Π

The hyperbolic vector field is given by

$$X_z^\sharp = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}.$$

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The hyperbolic moment map for the $\mathbb{R}_+$-action on Π is

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Coordinates

The $\mathbb{R}_+$-action on Π and the moment map associated to this action, induce a system of coordinates for Π given by

$$(u, v) = \left(\frac{\mathrm{Re}(z)}{\mathrm{Im}(z)}, \mathrm{Im}(z)\right) = \left(\frac{x}{y}, y\right),$$

where $z = x + iy \in \Pi$.

Contenido

Alvaro P. Raposo, Hans J. Weber, David E. Alvarez-Castillo, Mariana Kirchbach: *Romanovski polynomials in selected physics problems*, Central European Journal of Physics 5(3) (2007) 253-284, https://doi.org/10.2478/s11534-007-0018-5

Romanovski polynomials

Romanovski polynomials may be derived as the polynomial solutions of the ordinary differential equation

$$s(x)R''(x) + t(x)R'(x) + \Lambda R(x) = 0, \tag{2}$$

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where $s(x) = 1 + x^2$, $t(x) = 2bx + a$ and $a, b \in \mathbb{R}$.
The previous equation is a particular case of the hypergeometric differential equation

$$s(x)F''(x) + t(x)F'(x) + \Lambda F(x) = 0,$$

where F is a real function of a real variable in some open subset of the real line, $\Lambda \in \mathbb{R}$ is a corresponding eigenvalue, s is a polynomial of at most second order and t is a polynomial of at most first order.

Furthermore, for any $m \in \{0, 1, 2\ldots\}$, there exists a polynomial $r_m^{(a,b)}$ of degree m, together with a constant Λ_m which satisfies (2). The constant is given by

$$\Lambda_m = -m\left(t'(x) + \frac{1}{2}(m-1)s''(x)\right).$$

Consider the weight function

$$w^{(a,b)}(x) = \left(1+x^2\right)^{b-1} e^{-a\cot^{-1}x}. \tag{3}$$

The function $w^{(a,b)}(x)$ is associated with equation (2) since it is a solution of Pearson's differential equation

$$[s(x)w^{(a,b)}(x)]' = t(x)w^{(a,b)}(x).$$

Rodrigues formula

$$r_m^{(a,b)}(x) = N_m \frac{(-1)^m}{w^{(a,b)}(x)} \frac{d^m}{dx^m} \left[w^{(a,b)}(x) s(x)^m \right], \quad 0 \leq j \leq m, \tag{4}$$

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The Romanovski polynomials of degree 0,1 and 2 are

$$\frac{1}{N_0} r_0^{(a,b)}(x) = 1,$$

$$\frac{1}{N_1} r_1^{(a,b)}(x) = (2bx + a),$$

$$\frac{1}{N_2} r_2^{(a,b)}(x) = \left[(2b+1)(2b+2)\,x^2 + 2\,(2b+1)\,ax + \left(2b + a^2 + 2 \right) \right].$$

Finite orthogonality

If $r_j^{(a,b)}(x)$ and $r_m^{(a,b)}(x)$ are Romanovski polynomials of degree j and m, respectively, then

$$\int_{\mathbb{R}} w^{(a,b)}(x) r_j^{(a,b)}(x) r_m^{(a,b)}(x) dx = 0, \tag{5}$$

if and only if, $j + m < 1 - 2b$. Furthermore, for $m < -b$, the integral converges and we have that

$$\int_{\mathbb{R}} w^{(a,b)}(x) \left(r_m^{(a,b)}(x) \right)^2 dx = 1.$$

Contenido

We introduce the mapping $\kappa : \Pi \longrightarrow \Pi$ given by

$$\kappa : (x, y) \mapsto w = (u, v),$$

where $u = \frac{x}{y}$, $v = y$ and $z = x + iy$.

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where $u = \frac{x}{y}$, $v = y$ and $z = x + iy$.
Under the mapping κ we have

$$d\nu_\lambda(z) = v d\nu_\lambda(w). \tag{6}$$

Introduce the space $L_2(\Pi, d\eta_\lambda)$, where

$$d\eta_\lambda(w) = c_\lambda v^{\lambda+1} d\mu(w), \quad \lambda > -1.$$

Consider the unitary operator $U_0 : L_2(\Pi, d\nu_\lambda) \longrightarrow L_2(\Pi, d\eta_\lambda)$, defined as

$$(U_0 f)(w) = f(\kappa(w)),$$

and its adjoint operator

$$(U_0^* f)(z) = f(\kappa^{-1}(z))$$

Representing the upper half-plane Π in (u, v) coordinates we have the tensor decomposition

$$L_2(\Pi, d\eta_\lambda(w)) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, c_\lambda v^{\lambda+1} dv),$$

with $\lambda \in (-1, \infty)$ and c_λ given by (3).

Bergman Space

Now the image of the Bergman space under the operator U_0, $A_0(\Pi) = U_0(\mathcal{A}^2_\lambda(\Pi))$, is the subspace of $L_2(\Pi, d\eta_\lambda)$ which consists of all functions φ satisfying the equation

$$U_0 2 \frac{\partial}{\partial \bar{z}} U_0^{-1} \varphi = 0.$$

It follows that

$$U_0 2 \frac{\partial}{\partial \bar{z}} U_0^{-1} = BD, \tag{7}$$

where,

$$B = \frac{i}{v} \quad \text{and} \quad D = -(i+u)\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}.$$

$$DB = B(D-1). \tag{8}$$

Using (7) and (8) we have

$$\begin{aligned} U_0\left(2\frac{\partial}{\partial \overline{z}}\right)^n U_0^{-1}\varphi &= \left(U_0 2\frac{\partial}{\partial \overline{z}} U_0^{-1}\right)^n \varphi \\ &= (BD)^n \varphi \\ &= B^2(D-1)D\cdots BD\varphi \\ &\vdots \\ &= B^n(D-(n-1))\cdots(D-1)D\varphi. \end{aligned} \tag{9}$$

$A^2_{0,n,\lambda} = U_0(\mathcal{A}^2_{n,\lambda}(\Pi))$ is the closed subspace of $L_2(\Pi, d\eta_\lambda)$ that consists of all n-analytic functions φ in $L_2(\Pi, d\eta_\lambda)$ satisfying the equation

$$\tilde{D}\varphi = (D-(n-1))\cdots(D-1)D\varphi = 0.$$

Henceforth we denote by $A^2_{0,n} = A^2_{0,n,\lambda}$ and assume λ a fixed constant.
Introduce the unitary operator

$$U_1 = I \otimes M,$$

from $L_2\left(\mathbb{R}\right) \otimes L_2(\mathbb{R}_+, c_\lambda v^{\lambda+1} dv)$ onto $L_2\left(\mathbb{R}\right) \otimes L_2(\mathbb{R}, c_\lambda d\xi)$, where the Mellin transform $M : L_2(\mathbb{R}_+, v^{\lambda+1} dv) \longmapsto L_2(\mathbb{R})$ is given by

$$(M\psi)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} v^{-i\xi+\frac{\lambda}{2}} \psi(v) dv.$$

Since

$$M\left(v\frac{\partial}{\partial v}\right)M^{-1} = i\left(\xi + \left(\frac{\lambda}{2}+1\right)i\right)I.$$

We have

$$U_1\tilde{D}U_1^{-1} = \left[-(i+u)\frac{\partial}{\partial u} + i\left(\xi + \left(\frac{\lambda}{2}+1\right)i\right) - (n-1)\right]\cdots$$
$$\cdots\left[-(i+u)\frac{\partial}{\partial u} + i\left(\xi + \left(\frac{\lambda}{2}+1\right)i\right)\right].$$

Then, $A^2_{1,n} := U_1(A^2_{0,n})$ is the subspace of $L_2(\mathbb{R}) \otimes L_2(\mathbb{R}, c_\lambda d\xi)$ which consists of all functions $h \in L_2(\mathbb{R}) \otimes L_2(\mathbb{R}, c_\lambda d\xi)$ satisfying the equation

$$\left[-(i+u)\frac{\partial}{\partial u} + i\left(\xi + \left(\frac{\lambda}{2}+1\right)i\right) - (n-1)\right]\cdots$$
$$\cdots\left[-(i+u)\frac{\partial}{\partial u} + i\left(\xi + \left(\frac{\lambda}{2}+1\right)i\right)\right]h = 0.$$

The general solution of the previous equation has the following form

$$h(u,\xi) = \sum_{k=0}^{n-1} g_k(\xi)(i+u)^{i[\xi+(\lambda/2+1)i]-k}.$$

It will be convenient to rewrite $h(u,\xi)$ as follows

$$\begin{aligned} h(u,\xi) &= (i+u)^{i[\xi+(\lambda/2+1)i]-(n-1)} \sum_{k=0}^{n-1} g_k(\xi)(i+u)^{(n-1)-k} \\ &= (i+u)^{i[\xi+(\lambda/2+1)i]-(n-1)} \sum_{k=0}^{n-1} a_k(\xi)u^k. \end{aligned} \tag{10}$$

The functions $a_k(\xi)$ and $g_k(\xi)$ in (10), are adequate integrable functions such that $h(u,\xi)$ belongs to $A^2_{1,n}$.

With the Romanovski polynomials, the previous expression for $h(u,\xi)$ transforms to

$$\begin{aligned} h(u,\xi) &= (i+u)^{i[\xi+(\lambda/2+1)i]-(n-1)} \sum_{k=0}^{n-1} a_k(\xi) u^k \\ &= (i+u)^{i[\xi+(\lambda/2+1)i]-(n-1)} \sum_{k=0}^{n-1} h_k(\xi) r_k^{(a,b)}(u), \end{aligned}$$

where $a = -2\xi$, $b = -\frac{\lambda}{2} - n + 1$.

Let f be given by

$$f(u,\xi) = (i+u)^{i[\xi+(\lambda/2+1)i]-(n-1)} \sum_{k=0}^{n-1} f_k(\xi) r_k^{(a,b)}(u).$$

$$\langle h(u,\xi), f(u,\xi)\rangle = c_\lambda \sum_{k=0}^{n-1}\sum_{j=0}^{n-1} \int_{\mathbb{R}} h_k(\xi)\bar{f}_j(\xi) \int_{\mathbb{R}} r_k^{(a,b)}(u) r_j^{(a,b)}(u) w^{(a,b)}(u) du d\xi,$$

for the functions $h(u,\xi)$ and $f(u,\xi)$ to be in $A^2_{1,n}$, it is necessary that the functions $f_k(\xi)$ and $h_k(\xi)$ belong to $L_2(\mathbb{R})$.
For $k,j \leq n-1$ we have $k+j < 1-2b$, the orthogonality condition.

We introduce the unitary operator

$$U_2 : L_2(\mathbb{R}) \otimes L_2(\mathbb{R}, c_\lambda d\xi) \longrightarrow \int_{\mathbb{R}}^{\oplus} L_2(\mathbb{R}, d\omega_{a,b}) c_\lambda d\xi$$
$$g(u,\xi) \longmapsto g(u,\xi)(i+u)^{-i[\xi+(\lambda/2+1)i]+(n-1)},$$

where

$$d\omega_{a,b} = (1+u^2)^{-\lambda/2-n} e^{-2\xi cot^{-1}(u)} du = w^{(a,b)}(u) du,$$

with $w^{(a,b)}(u)$ the Romanovski weight function.
Moreover, the adjoint operator is given by

$$U_2^* : \int_{\mathbb{R}}^{\oplus} L_2(\mathbb{R}, d\omega_{a,b}) c_\lambda d\xi \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}, c_\lambda d\xi)$$
$$f(u,\xi) \longmapsto f(u,\xi)(i+u)^{i[\xi+(\lambda/2+1)i]-(n-1)}.$$

$A^2_{2,n} = U_2(A^2_{1,n})$ this space corresponds to the functions of the form

$$f(u,\xi) = \sum_{k=0}^{n-1} f_k(\xi) r_k^{(-2\xi,b)}(u).$$

Theorem

The unitary operator $U = U_2 U_1 U_0$ *gives an isometric isomorphism from the space* $L_2(\Pi, d\nu_\lambda)$ *onto* $\int_{\mathbb{R}}^{\oplus} L_2(\mathbb{R}, d\omega_{a,b}) c_\lambda d\xi$, *under which*

1. *the weighted poly-Bergman space* $\mathcal{A}^2_{n,\lambda}(\Pi)$ *is mapped onto* $A^2_{2,n}$, *where every function in* $A^2_{2,n}$ *has the form*
$$f(u,\xi) = \sum_{k=1}^{n} f_k(\xi) r_{k-1}^{(-2\xi,b)}(u),$$
where $b = -\frac{\lambda}{2} - n + 1$.
2. *the poly-Bergman projection* $B_{\Pi,n,\lambda}$ *is unitarily equivalent to* $P_{n,\lambda} = U B_{\Pi,n,\lambda} U^*$ *the projection from* $\int_{\mathbb{R}}^{\oplus} L_2(\mathbb{R}, d\omega_{a,b}) c_\lambda d\xi$ *onto the space* $A^2_{2,n}$, *which is given by*
$$(P_{n,\lambda} f)(u,\xi) = \sum_{k=1}^{n} r_{k-1}^{(-2\xi,b)}(u) \int_{\mathbb{R}} f(v,\xi) r_{k-1}^{(-2\xi,b)}(v) dw_{a,b}(v).$$

Introduce now the isometric embedding

$$R_{0,n} : (L_2(\mathbb{R}))^n \longrightarrow \int_{\mathbb{R}}^{\oplus} L_2(\mathbb{R}, d\omega_{a,b}) c_\lambda d\xi$$

by the rule

$$(R_{0,n}f)(u,\xi) = \sum_{k=0}^{n-1} f_k(\xi) r_k^{(-2\xi,b)}(u) = f(\xi)(\mathbf{r}^{(-2\xi,b)}(u))^t$$

where $f = (f_0, \ldots, f_{n-1})$, $f_k(\xi) \in L_2(\mathbb{R})$ for $k = 0, \ldots, n-1$, and

$$\mathbf{r}^{(-\mathbf{2}\xi,\mathbf{b})}(u) = \left(\; r_0^{(-2\xi,b)}(u), \quad r_1^{(-2\xi,b)}(u), \quad \cdots \quad , r_{n-1}^{(-2\xi,b)}(u) \; \right).$$

The image of $R_{0,n}$ is the space $A^2_{2,n}$.

The adjoint operator

$$R_{0,n}^{*} : \int_{\mathbb{R}}^{\oplus} L_2(\mathbb{R}, d\omega_{a,b}) c_\lambda d\xi \longrightarrow (L_2(\mathbb{R}))^n$$

has the form

$$\begin{aligned}
&(R_{0,n}^{*}\varphi)(\xi) \\
&= \left(\int_{\mathbb{R}} \varphi(u,\xi) r_0^{(-2\xi,b)}(u) d\omega_{a,b}(u), \ldots, \int_{\mathbb{R}} \varphi(u,\xi) r_{n-1}^{(-2\xi,b)}(u) d\omega_{a,b}(u) \right) \\
&= \int_{\mathbb{R}} \varphi(u,\xi) \mathbf{r}^{(-2\xi,b)}(u) d\omega_{a,b}(u) \qquad (11)
\end{aligned}$$

then

$$R_{0,n}^{*} R_{0,n} = I : (L_2(\mathbb{R}))^n \longrightarrow (L_2(\mathbb{R}))^n$$

$$R_{0,n} R_{0,n}^{*} = P_{n,\lambda} : \int_{\mathbb{R}}^{\oplus} L_2(\mathbb{R}, d\omega_{a,b}) c_\lambda d\xi \longrightarrow \int_{\mathbb{R}}^{\oplus} L_2(\mathbb{R}, d\omega_{a,b}) c_\lambda d\xi.$$

Theorem

The operator $R_n = R_{0,n}^* U$ *maps* $L_2(\Pi, d\nu_\lambda)$ *onto* $(L_2(\mathbb{R}))^n$*, and the restriction*

$$R_n|_{\mathcal{A}^2_{n,\lambda}(\Pi)} : \mathcal{A}^2_{n,\lambda}(\Pi) \longrightarrow (L_2(\mathbb{R}))^n,$$

is an isometric isomorphism. The adjoint operator

$$R_n^* = U^* R_{0,n} : (L_2(\mathbb{R}))^n \longrightarrow \mathcal{A}^2_{n,\lambda}(\Pi) \subset L_2(\Pi, d\nu_\lambda)$$

is the isometric isomorphism from $(L_2(\mathbb{R}))^n$ *onto the subspace* $\mathcal{A}^2_{n,\lambda}(\Pi)$*. Furthermore,*

$$R_n^* R_n = B_{\Pi,n,\lambda} : L_2(\Pi, d\nu_\lambda) \longrightarrow \mathcal{A}^2_{n,\lambda}(\Pi)$$
$$R_n R_n^* = I : (L_2(\mathbb{R}))^n \longrightarrow (L_2(\mathbb{R}))^n,$$

where $B_{\Pi,n,\lambda}$ *is the Bergman projection of* $L_2(\Pi, d\nu_\lambda)$ *onto* $\mathcal{A}^2_{n,\lambda}(\Pi)$*.*

Let $a(z) = \frac{x}{y}$ be a function in $L_\infty(\Pi)$ depending on $\mu(z) = \frac{x}{y}$, the hyperbolic moment map for the $\mathbb{R}_+$-action on Π. We shall say that $a(z)$ is a homogeneous symbol. The Toeplitz operator acting on $\mathcal{A}^2_{n,\lambda}(\Pi)$ with symbol $a(z)$ is the operator denoted by

$$\mathrm{T}_a : \varphi \in \mathcal{A}^2_{n,\lambda}(\Pi) \longmapsto B_{\Pi,n,\lambda}(a\varphi) \in \mathcal{A}^2_{n,\lambda}(\Pi). \tag{12}$$

Theorem

For any $a(z) \in L_\infty(\Pi)$, the Toeplitz operator T_a acting on $\mathcal{A}^2_{n,\lambda}(\Pi)$ is unitary equivalent to the matrix multiplication operator $\gamma^{n,a}(\xi)I = R_n \mathrm{T}_a R_n^*$ acting on $(L_2(\mathbb{R}))^n$, where the matrix-valued function $\gamma^{n,a} = (\gamma^{n,a}_{i,j})$ is given by

$$\gamma^{n,a}(\xi) = \int_{\mathbb{R}} a(u)(\mathbf{r}^{(-2\xi,b)}(u))^t \mathbf{r}^{(-2\xi,b)}(u) d\omega_{a,b}(u). \tag{13}$$

Gracias