# A Spectral Theorem approach to commutative algebras generated by Bergman-Toeplitz operators

Grigori Rozenblum

Mexico, Sept. 2022

Part 3. Toeplitz operators on the ball.

# The Bergman space on the ball $\mathbb{B}^{\mathbf{n}}$. Notation.

$\mathbb{B}^{\mathbf{n}} = \{z = (z_1, ..., z_{\mathbf{n}}) \in \mathbb{C}^{\mathbf{n}} : \ |z|^2 \equiv |z_1|^2 + ... + |z_{\mathbf{n}}|^2 < 1\}$. Given a multi-index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_{\mathbf{n}}) \in \mathbb{Z}_+^{\mathbf{n}}$ we will use the standard notation,

$$\begin{aligned} |\alpha| &= \alpha_1 + \alpha_2 + ... + \alpha_{\mathbf{n}}, \\ \alpha! &= \alpha_1!\, \alpha_2! \cdots \alpha_{\mathbf{n}}!, \\ z^\alpha &= z_1^{\alpha_1}\, z_2^{\alpha_2} \cdots z_{\mathbf{n}}^{\alpha_{\mathbf{n}}}, \end{aligned}$$

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$dV = dx_1 dy_1 ... dx_{\mathbf{n}} dy_{\mathbf{n}}$ the standard Lebesgue measure in $\mathbb{C}^{\mathbf{n}}$,$dv_\lambda(z) = c_\lambda\, (1 - |z|^2)^\lambda\, dV(z)$, $\lambda > -1$ , the normalized weighted measure.

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$$e_\alpha = e_{\alpha,\lambda}(z) := e_{\alpha,\lambda,(\mathbf{n})}(z) = \omega_{\lambda,\alpha,(\mathbf{n})} z^\alpha, \quad \text{with} \quad \alpha \in \mathbb{Z}_+^{\mathbf{n}}.$$

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Note: The normalization factor depends on $\mathbf{n}$ and $\lambda$.
For a fixed $j \in [1, \mathbf{n}]$, we denote by $\widetilde{z}_j$ the collection of complementary variables, $(z_{j'})$, $j' \neq j$.

$\mathcal{A}_\lambda(\mathbb{B}^{\mathbf{n}})$ consists of analytic functions $f(z) = f(z_1, \ldots, z_{\mathbf{n}})$ for which the Fourier coefficients

$$f_\alpha = \langle f, e_{\alpha,\lambda} \rangle_{\mathcal{H}(\mathbb{B}^{\mathbf{n}})}$$

satisfy $\sum_{\alpha \in \mathbb{Z}_+^{\mathbf{n}}} |f_\alpha|^2 < \infty$.

# Main results

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On the other hand we associate with this type a system of commuting self-adjoint operators $\mathbf{V}$ so that the the joint spectral measure is block-diagonal with respect to the same structure.
Thus, the algebra generated by Toeplitz operators with this type of homogeneity is isomorphic to an algebra of functions of this system of operators.

# Type 1. Toeplitz operators with radial symbols

Radial symbols will act as a kind of building blocks for more general spectral calculus. The extension of the case of the disk $\mathbb{D} = \mathbb{B}^1$. The system of operators consists of only one operator $V$. Consider the dimension $k \leq \mathbf{n}$.

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$$V = z_1 \frac{\partial}{\partial z_1} + \cdots + z_k \frac{\partial}{\partial z_k} = \frac{1}{\imath} \frac{\partial}{\partial \theta_1} + \cdots + \frac{1}{\imath} \frac{\partial}{\partial \theta_k},$$

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$$\mathcal{D}(V) = \left\{ f = \sum_{\alpha \in \mathbb{Z}_+^k} f_\alpha e_{\alpha,(k)} \in \mathcal{A}_\lambda(\mathbb{B}^k) : \sum_{\alpha \in \mathbb{Z}_+^k} |\alpha|^2 |f_\alpha|^2 < \infty \right\}.$$

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$V$ is diagonal in the basis $\{e_\alpha\}$ and it acts upon such element in the basis as $Ve_\alpha = |\alpha| e_\alpha$. Therefore, each nonnegative integer $\ell \in \mathbb{Z}_+$ is an eigenvalue of $V$, and the multiplicity of $\ell$ equals $\mathbf{d}_\ell = \binom{\ell+k-1}{k-1}$.

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# The spectral measure of $V$.

Denote $\mathbf{P}_\ell$ the orthogonal (in $\mathcal{A}(\mathbb{B}^k)$) projection onto $\mathscr{L}_\ell$. It can be expressed as

$$\mathbf{P}_\ell f = \sum_{|\alpha|=\ell} \langle f, e_\alpha \rangle e_\alpha,$$

where $\langle ., . \rangle$ denotes the scalar product in the Hilbert space $\mathcal{H}_\lambda(\mathbb{B}^k)$. $\mathbf{P}_\ell$ is an integral operator $(\mathbf{P}_\ell f)(z) = \int_{\mathbb{B}^k} f(\zeta)\mathbf{P}_\ell(z,\zeta)dv_\lambda(\zeta)$ with the (degenerate) integral kernel $\mathbf{P}_\ell(z,\zeta) = \frac{1}{\ell!} C_{\lambda,\alpha,k}\langle z, \bar{\zeta}\rangle^\ell$. The spectral measure for $V$ is given by

$$E_V(\Delta) = \sum_{\ell \in \Delta} \mathbf{P}_\ell, \tag{0.1}$$

with the corresponding resolution of identity $\mathbf{E}_V(\eta) = \sum_{\ell \leq \eta} \mathbf{P}_\ell$. Thus, for any $f \in \mathcal{A}(\mathbb{B}^k)$, the measure $\rho_f(\Delta) = \langle E_V(\Delta)f, f\rangle$ is supported at integer points.

For an $E_V$-measurable complex-valued function $\varphi(\eta)$, $\eta \in \mathbb{R}^1$, the operator $\varphi(V)$ is defined as

$$\varphi(V) = \sum_{\ell \in \mathbb{Z}_+} \varphi(\ell) \mathbf{P}_\ell,$$

$$\mathcal{D}(\varphi(V)) = \{f = \sum f_\alpha e_{\alpha,(k)} \in \mathcal{A}(\mathbb{B}^k) : \sum_\ell |\varphi(\ell)|^2 \sum_{|\alpha|=\ell} |f_\alpha|^2 < \infty\}.$$

On this domain, the operator $\varphi(V)$ is normal; if $\varphi$ has real values at all integer points, $\varphi(V)$ is self-adjoint; it is bounded iff $\varphi$ is a bounded function on $\mathbb{Z}_+$.

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For different functions $\varphi_1, \varphi_2$ the operators $\varphi_1(V), \varphi_2(V)$ commute $\mathcal{J} : \varphi \mapsto \varphi(V)$ is an isomorphism of the $C^*$ algebra of bdd sequences onto a commutative algebra of operators in $\mathcal{A}_\lambda(\mathbb{B}^k)$.

# Radial Toeplitz operators

. $\varphi = \mathbf{1}$, $\mathbf{1}(\ell) = 1$ for all $\ell \in \mathbb{Z}_+$, gives $\mathbf{1}(V) = \mathbf{Id}_{\mathcal{A}_\lambda(\mathbb{B}^k)}$, the identity operator in $\mathcal{A}_\lambda(\mathbb{B}^k)$. If $f \in \mathcal{H}_\lambda(\mathbb{B}^k)$ and $f$ is orthogonal to all functions in $\mathcal{A}_\lambda(\mathbb{B}^k)$, we have $\mathbf{1}(V)f = 0$ since each $\mathbf{P}_\ell f$ equals zero. Therefore, $\mathbf{1}(V)$ is nothing else but the Bergman projection $\mathbb{P} \equiv \mathbb{P}_\lambda : \mathcal{H}_\lambda(\mathbb{B}^k) \to \mathcal{A}_\lambda(\mathbb{B}^k)$.

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Let $a(r) = a(|z|) \in L_\infty(0,1)$ radial symbol on the ball $\mathbb{B}^k$. The Toeplitz operator $\mathbf{T}_a : \mathbf{T}_a f = \mathbb{P} a f$ in $\mathcal{A}_\lambda(\mathbb{B}^k)$ Checking the action of $\mathbf{T}_a$ on $f = e_\alpha$.

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Namely, with the radial symbol $a(r)$, we associate the function $\varphi_a$, $\varphi_a(\ell) = \gamma_{\ell,a}, \ell \in \mathbb{Z}_+$,

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**Theorem** For a radial symbol $a$, the Toeplitz operator $\mathbf{T}_a$ in $\mathcal{A}_\lambda(\mathbb{B}^k)$ is a function of the operator $V$,

$$\mathbf{T}_a = \varphi_a(V).$$

The operator $\mathbf{T}_a$ is bounded iff the function $\varphi_a$ is bounded on $\mathbb{Z}_+$; $\mathbf{T}_a$ is compact iff $\varphi_a(\ell) \to 0$ as $\ell \to \infty$.
Similarly to the disk case, the relation $a \Leftrightarrow \varphi_a$ extends to a wide class of unbounded and distributional symbols $a$. All other 'disk' results carry over.

# Toeplitz operators with separately radial symbols

Another extreme class of symbols, radial with respect to each of variables $z_j$ separately, with all other variables fixed. A measurable function $a(z)$, $z \in \mathbb{B}^{\mathbf{n}}$, is called *separately radial* if it depends only on $r_1, \ldots, r_{\mathbf{n}}$, where $r_j = |z_j|$. A basic function $e_\alpha = e_{\alpha,\mathbf{n}}$ is not separately radial, but the one-dimensional subspace spanned by $e_\alpha$ is invariant with respect to the action of rotations with respect to each $z_j$ variable, and this property pertains after the multiplication by a separately radial function $a(z)$, as well as the orthogonality,

$$\langle e_\alpha, a(z) e_{\alpha'} \rangle = 0, \; \alpha \neq \alpha'.$$

Therefore, for a separately radial symbol $a$, the Toeplitz operator $\mathbf{T}_a : f \mapsto \mathbb{P}af$ in $\mathcal{A}_\lambda(\mathbb{B}^{\mathbf{n}})$ is diagonal in the basis $e_\alpha$,

$$\mathbf{T}_a e_\alpha = \gamma_a e_\alpha = \gamma_{\alpha,\mathrm{sep}} e_\alpha$$

where $\gamma_a(\alpha) \equiv \gamma_{a,\mathrm{sep}}(\alpha) =< a e_\alpha, e_\epsilon >$
(...an explicit expression...)

For a fixed multi-index $\mathbf{s} = (s_1, \ldots, s_{\mathbf{n}}) \in \mathbb{Z}_+^{\mathbf{n}}$, we define the one-dimensional subspace $H_{\mathbf{s}}$ spanned by $e_{\mathbf{s}}$. We also define, for a given $\ell \in \mathbb{Z}_+$ and a fixed $j$, the *infinite-dimensional* subspace $H_\ell^{(j)} \subset \mathcal{A}_\lambda(\mathbb{B}^{\mathbf{n}})$ as

$$H_\ell^{(j)} = \overline{\check{H}_\ell^{(j)}} \equiv \overline{\mathrm{span}}\,\{e_\alpha : \ \alpha_j = \ell\}.$$

It is important to note here that the non-closed space $\check{H}_\ell^{(j)}$ consists of all polynomials in $z$ variables containing $z_j$ exactly in the power $\ell$, $f(z) = z_j^\ell h(\widetilde{z}_j)$, in other words, it consists of polynomials which are proportional to $z_j^\ell$ for any fixed $\widetilde{z}_j$. After the closure in the norm of $\mathcal{A}_\lambda(\mathbb{B}^{\mathbf{n}})$, this structure is modified in the following way. We calculate the norm in $\mathcal{A}_\lambda(\mathbb{B}^{\mathbf{n}})$ of a function $f(z) = z_j^\ell h(\widetilde{z}_j)$. the result:

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The space $H_\ell^{(j)}$ consists of functions of the form $f(z) = z_j^\ell h(\widetilde{z}_j)$, with $h$ belonging to the Bergman space $\mathcal{A}_{\lambda+\ell+1}(\mathcal{B}^{\mathbf{n}-1})$. Moreover, the norms of $f$ in $\mathcal{A}_\lambda(\mathcal{B}^{\mathbf{n}})$ and $h$ in $\mathcal{A}_{\lambda+\ell+1}(\mathcal{B}^{\mathbf{n}-1})$ are equivalent.

The orthogonal projections, $\mathbf{P_s} : \mathcal{H} \longrightarrow H_{\mathbf{s}}$ (a rank one projection) and $\mathbf{P}_\ell^{(j)} : \mathcal{H} \longrightarrow H_\ell^{(j)}$, so that $\mathbf{P_s} = \prod_{j=1}^{\mathbf{n}} \mathbf{P}_{s_j}^{(j)}$. For each $j, \ell$, the monomials $e_\alpha$ with $\alpha_j = \ell$ form an orthonormal basis in $H_\ell^{(j)}$. Therefore the projection $\mathbf{P}_\ell^{(j)}$ can be written as

$$(\mathbf{P}_\ell^{(j)}) f(z_j, \widetilde{z_j}) = \sum_{\alpha_j = \ell} \langle f, e_\alpha \rangle e_\alpha(z).$$

The Bergman space $\mathcal{A}_\lambda(\mathbb{B}^{\mathbf{n}})$ splits into the orthogonal sum in two ways:

$$\mathcal{A}_\lambda(\mathbb{B}^{\mathbf{n}}) = \bigoplus_{\mathbf{s} \in \mathbb{Z}_+^{\mathbf{n}}} H_{\mathbf{s}} \qquad \text{and} \qquad \mathcal{A}_\lambda(\mathbb{B}^{\mathbf{n}}) = \bigoplus_{\ell \in \mathbb{Z}_+} H_\ell^{(j)}.$$

In the first decomposition we split the Bergman space into the direct sum of one-dimensional subspaces, in the second one the subspaces consist of functions having a fixed homogeneity in the variable $z_j$ (and analytic in all remaining variables.)

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The spectral measures of the self-adjoint operators $\mathbf{V} : (V_j, j \leq \mathbf{n})$ commute, with $\mathbb{Z}_+^{\mathbf{n}}$ being the joint spectrum. Therefore, the joint spectral measure $E$ is supported on the integer lattice $\mathbb{Z}_+^{\mathbf{n}}$, while the set $\mathbb{R}^{\mathbf{n}} \setminus \mathbb{Z}_+^{\mathbf{n}}$ has zero $E$-measure. The general spectral theory for systems of commuting operators, presented in Part 1 applies to $\mathbf{V} = (V_1, \ldots, V_{\mathbf{n}})$. Each multi-index $\mathbf{s} = (s_1, \ldots, s_{\mathbf{n}})$, is a point of joint spectrum of $\mathbf{V}$. The spectral projection: $P_{\mathbf{s}} = \prod P_{s_j}$; **It has rank one.**

# The Functional Calculus:

**Proposition** Given an $E$-measurable function $\varphi(\boldsymbol{\eta})$ on $\mathbb{R}^{\mathbf{n}}$, the operator

$$\varphi(\mathbf{V}) \equiv \varphi(V_1, \ldots, V_n) :=$$
$$\int_{\mathbb{R}^{\mathbf{n}}} \varphi(\eta_1, \ldots, \eta_{\mathbf{n}})\, dE(\eta_1, \ldots, \eta_{\mathbf{n}}) = \sum_{\mathbf{s} \in \mathbb{Z}_+^{\mathbf{n}}} \varphi(\mathbf{s}) P_{\mathbf{s}}$$

is well defined and normal on its domain

$$\mathcal{D}_\varphi = \left\{ f \in \mathcal{A}_\lambda(\mathbb{B}^{\mathbf{n}}) : \int_{\mathbb{R}^{\mathbf{n}}} |\varphi(\eta_1, \ldots, \eta_n)|^2 \, d\langle E(\eta_1, \ldots, \eta_{\mathbf{n}}) f, f \rangle < \infty \right\}.$$

Moreover the operator $\varphi(V_1, \ldots, V_{\mathbf{n}})$ is bounded, and thus defined on the whole $\mathcal{H}$, if and only if the function $\varphi$ is $E$-essentially bounded.

The set of all operators $\{\varphi(\mathbf{V})\}$, defined by (the classes of equivalency) of $E$-measurable essentially bounded functions $\varphi$ with $\boldsymbol{\varphi} = \{\varphi(\mathbf{s})\}_{\mathbf{s}\in\mathbb{Z}_+^{\mathbf{n}}} \in \ell_\infty(\mathbb{Z}_+^{\mathbf{n}})$ constitutes the algebra $\mathcal{R}$ of bounded mutually commuting operators in $\mathcal{A}_\lambda(\mathbb{B}^{\mathbf{n}})$, and the mapping

$$\mathcal{J}_{\mathbf{V}} : \varphi \longmapsto \varphi(\mathbf{V}) = \sum_{\mathbf{s}\in\mathbb{Z}_+^{\mathbf{n}}} \varphi(\mathbf{s}) P_{\mathbf{s}}.$$

defines the isomorphism

$$\mathcal{J}_{\mathbf{V}} : \ell_\infty(\mathbb{Z}_+^{\mathbf{n}}) \longrightarrow \mathcal{R}$$

of commutative $C^*$-algebras.

# Toeplitz operators with separately radial symbols

$a(z) = a(r_1, \ldots, r_{\mathbf{n}})$ is a **separately radial symbol**. The Toeplitz operator $\mathbf{T}_a$, with symbol $a = a(r_1, r_2, \ldots, r_{\mathbf{n}})$ is diagonal in the basis $e_\alpha$

$$\mathbf{T}_a = \sum_{\mathbf{s} \in \mathbb{Z}_+^{\mathbf{n}}} \gamma_a(\mathbf{s}) \mathbf{P}_{\mathbf{s}},$$

where the scalars $\gamma_a(\mathbf{s})$ are given by an explicit expression. Therefore,

$$\mathbf{T}_a = \varphi_a(V_1, V_2, \ldots, V_{\mathbf{n}}),$$

where $\varphi_a$ belongs to the equivalence class of essentially bounded $E$-measurable functions, defined by the sequence $\boldsymbol{\varphi}_a = \{\varphi_a(\mathbf{s})\}_{\mathbf{s} \in \mathbb{Z}_+^{\mathbf{n}}}$, with $\varphi_a(\mathbf{s}) = \gamma_a(\mathbf{s})$.

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$$\mathbf{T}_a = \sum_{\mathbf{s} \in \mathbb{Z}_+^{\mathbf{n}}} \varphi_a(\mathbf{s}) \mathbf{P}_{\mathbf{s}} = \sum_{\mathbf{s} \in \mathbb{Z}_+^{\mathbf{n}}} \gamma_a(\mathbf{s}) \mathbf{P}_{\mathbf{s}}.$$

As before, $a$ bounded $\Rightarrow$ $\mathbf{T}_a$ is bounded, but not only for bounded $a$.

# Operators with quasi - radial symbols

A combination of the previous two ones: radial in separate groups of variables. For the set of such symbols, the classes considered in two previous sections serve as extreme cases.
Let $\mathbf{k} = (k_1, ..., k_m)$ be a *tuple* of positive integers whose sum is equal to $\mathbf{n}$: $k_1 + ... + k_m = \mathbf{n}$; in other words, it is a *partition* of $\mathbf{n}$ into positive integers. The *length* $m$ of such a partition may, vary from 1, for $\mathbf{k} = (\mathbf{n})$, to $\mathbf{n}$, for $\mathbf{k} = (1, ..., 1)$.

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$\mathbf{k} = (k_1, ..., k_m)$, we arrange the coordinates of $z \in \mathbb{B}^{\mathbf{n}}$ in $m$ groups, having, correspondingly, has $k_j$, $j = 1, ..., m$, entries

$$z_{(1)} = (z_{1,1}, ..., z_{1,k_1}) \in \mathbb{C}^{k_1},\ z_{(2)} = (z_{2,1}, ..., z_{2,k_2}) \in \mathbb{C}^{k_2},\ ...,$$
$$z_{(m)} = (z_{m,1}, ..., z_{m,k_m}) \in \mathbb{C}^{k_m},$$

So, $\mathbb{B}^{\mathbf{n}}$ is:

$$\mathbb{B}^{\mathbf{n}} \ni z = (z_1, z_2, \ldots, z_{\mathbf{n}}) = (z_{(1)}, z_{(2)}, \ldots, z_{(m)}), \sum |z_{(j)}|^2 < 1.$$

With a group $z_{(j)}$ fixed, the *complementing* group of variables will be denoted by

$$\widetilde{z_{(j)}} = (z_{(1)}, \ldots, z_{(j-1)}, z_{(j-1)}, \ldots, z_{(m)}),$$

so, with an obvious permutations of variables $z = (z_{(j)}, \widetilde{z_{(j)}})$. In the extreme case $\mathbf{k} = (1, 1, \ldots, 1)$, the separately radial case, $\widetilde{z_j} = \widetilde{z_{(j)}}$. (Note the subtlety in notation: the subscript $j$ (as in $z_j$, $\widetilde{z_j}$, $V_j$, etc.) is used for denoting a single complex variable or some related object, while the subscript $(j)$ is used to denote the tuple of variables and related objects, as in $z_{(j)}$ etc.)

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A measurable function $a = a(z)$, $z \in \mathbb{B}^{\mathbf{n}}$, will be called $\mathbf{k}$-*quasi-radial* if it depends only on $r_{(1)}$, ..., $r_{(m)}$, where

$$r_{(j)}^2 = |z_{(j)}| = \sqrt{|z_{j,1}|^2 + \ldots + |z_{j,k_j}|^2}, \qquad j = 1, \ldots, m.$$

With this definition, such symbols can vary from *separately radial*, $a = a(|z_1|, \ldots, |z_{\mathbf{n}}|)$, if $\mathbf{k} = (1, ..., 1)$, to *radial*, $a = a(|z|)$, if $\mathbf{k} = (\mathbf{n})$. These cases have been demonstrated above.

$\mathbf{T}_a$ in $\mathcal{A}_\lambda(\mathbb{B}^{\mathbf{n}})$, with $\mathbf{k}$-quasi-radial symbol $a = a(r_1, \ldots, r_m)$, acts diagonally upon the basis elements $e_\alpha(z)$ as follows

$$\mathbf{T}_a\, e_\alpha = \gamma_{a,\mathbf{k},\lambda}(\alpha)\, e_\alpha, \qquad \alpha \in \mathbb{Z}_+^{\mathbf{n}},$$

where $\gamma_{a,\mathbf{k},\lambda}(\alpha)$ is given by some terrible integral.
Given a tuple $\mathbf{s} = (s_1, \ldots, s_m) \in \mathbb{Z}_+^m$, we define the finite dimensional subspace $H_s$ of $\mathcal{A}$ as

$$H_{\mathbf{s}} = \operatorname{span}\{e_\alpha : \ |\alpha_{(j)}| = s_j, \quad j = 1, \ldots, m\}.$$

This subspace consists of functions which are, for each $j = 1, \ldots, m$, homogeneous polynomials of order $s_j$ in the group of variables $z_{(j)}$.

With $\ell \in \mathbb{Z}_+$, for any fixed $j$, we define also the *infinite dimensional* subspace $H_\ell^{(j)}$ in $\mathcal{A}$ as

$$H_\ell^{(j)} = \overline{\mathrm{span}}\,\{e_\alpha : \ |\alpha_{(j)}| = \ell\}.$$

The latter space is the closure in $\mathcal{A}_\lambda(\mathbb{B}^{\mathbf{n}})$ of the space of polynomials which are homogeneous order $\ell$ in the group of variables $z_{(j)}$. With this definition, $H_{\mathbf{s}} = \bigcap_{j=1,\ldots,m} H_{s_j}^{(j)}$. We introduce the corresponding orthogonal projections

$$\mathbf{P}_{\mathbf{s}} : \mathcal{A} \longrightarrow H_{\mathbf{s}} \qquad \text{and} \qquad P_\ell^{(j)} : \mathcal{A} \longrightarrow H_\ell^{(j)}, \quad ; \quad \mathbf{P}_{\mathbf{s}} = \prod_{j=1,\ldots,m} P_{s_j}^{(j)}.$$

With such notation, the Bergman space $\mathcal{A}$ splits into the orthogonal sum in two ways:

$$\mathcal{A} = \bigoplus_{\mathbf{s}\in\mathbb{Z}_+^m} H_{\mathbf{s}} \qquad \text{and} \qquad \mathcal{A} = \bigoplus_{\ell\in\mathbb{Z}_+} H_\ell^{(j)}.$$

We see that in the second case, each summand consists of functions having the fixed homogeneity order in variables of $z_{(j)}$ and analytic in the remaining variables.

$$\mathbf{T}_a = \sum_{\mathbf{s}\in\mathbb{Z}_+^m} \gamma_{a,\mathbf{k},\lambda}(\mathbf{s})\mathbf{P}_{\mathbf{s}},$$

understood in the sense of the strong convergence. We group them, correspondingly to the radial structure.

For each $j = 1, \ldots, m$, we introduce the rotation operator along the group of variables $z_{(j)}$:

$$V_{(j)} = z_{j,1}\frac{\partial}{\partial z_{j,1}} + \ldots + z_{j,k_j}\frac{\partial}{\partial z_{j,k_j}} = \frac{1}{\imath}\frac{\partial}{\partial \theta_{j,1}} + \ldots + \frac{1}{\imath}\frac{\partial}{\partial \theta_{j,k_j}},$$

where $z_{j,\ell} = |z_{j,\ell}|e^{\imath\theta_{j,\ell}}$, which is self-adjoint being defined on the natural domain.

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where $z_{j,\ell} = |z_{j,\ell}|e^{\imath\theta_{j,\ell}}$, which is self-adjoint being defined on the natural domain.

The operator $V_{(j)}$ admits the representation

$$V_{(j)} = \sum_{\ell \in \mathbb{Z}_+} \ell P_\ell^{(j)},$$

understood in the sense of the strong convergence.

For each $j$ the operator $V_{(j)}$ acts only upon variables $z_l$ entering in the partition $z_{(j)}$. Therefore, these operators commute. For each fixed $j$, the operator $V_{(j)}$ has spectral structure similar to the one we described for radial operators for a ball of dimension $k = k_j$; the spectrum of $V_{(j)}$ consists of isolated points $\ell \in \mathbb{Z}_+$. Namely, each eigenfunction of $V_{(j)}$ corresponding to the eigenvalue $\ell$ has the form

$$f_\ell^{(j)}(z) = z_{(j)}^{\alpha_{(j)}} h(\widetilde{z_{(j)}})$$

The spectral measure $E$ for $\mathbf{V} = (V_1, \ldots, V_m)$, is supported on the integer lattice $\mathbb{Z}_+^m$,

$$V_{(j)} = \int_{\mathbb{R}^m} \eta_j d\mathcal{E}(\eta_1, ..., \eta_m).$$

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The Functional Calculus implies:
Given an $E$-measurable function $\varphi$ on $\mathbb{R}^m$, the operator

$$\varphi(\mathbf{V}) = \varphi(V_1, \ldots, V_m) := \int_{\mathbb{R}^m} \varphi(\eta_1, \ldots, \eta_m)\, d\mathcal{E}(\eta_1, \ldots, \eta_m)$$

is well defined and normal on its domain
Moreover the operator $\varphi(\mathbf{V})$ is bounded, and thus defined on the whole $\mathcal{A}_\lambda(\mathbb{B}^{\mathbf{n}})$, if and only if the function $\varphi$ is $E$-essentially bounded. Operators $\varphi(\mathbf{V})$ commute.

$$Op_E : \ell_\infty(\mathbb{Z}_+^m) \longrightarrow \mathcal{R}^{\mathbf{k}}$$

is an isomorphism of the $C^*$-algebras.

# Toeplitz operators:

The Toeplitz operator $\mathbf{T}_a$, with $\mathbf{k}$-quasi-radial symbol $a = a(r_1, r_2, \ldots, r_)$, admits the representation

$$\mathbf{T}_a = \sum_{\mathbf{s} \in \mathbb{Z}_+^m} \gamma_{a,\mathbf{k},\lambda}(\mathbf{s}) \mathbf{P}_{\mathbf{s}} = \varphi_a(\mathbf{V}),$$

# Discussion

1. We find a different construction of commutative algebras of Toeplitz operators, based upon the spectral theory of families of commuting operators. 2. We associate a spectral function with each Toeplitz operator; the algebra of spectral functions is commutative, unlike the algebra of symbols. 3. The construction extends to other commutative algebras of Toeplitz operators, where thew spectrum may contain an essential component.