Localization and Toeplitz operators in Fock-Segal-Bargmann spaces

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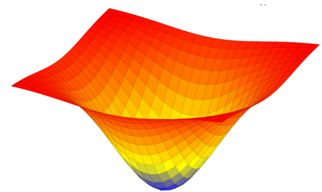

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Introduction: Fock-Segal-Bargmann space and Toeplitz operators

Notation

We will use the following standard notation:

- $z = x + iy = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$;
- the usual notion of the complex conjugation $\overline{z} = (\overline{z_1}, \overline{z_2}, \ldots, \overline{z_n})$;
- for $z, w \in \mathbb{C}^n$ $z \cdot w = \sum_{k=1}^{n} z_k w_k$;
- $z^2 = z \cdot z = \sum_{k=1}^{n} z_k^2$;
- $|z|^2 = z \cdot \overline{z} = \sum_{k=1}^{n} |z_k|^2$.

Consider the space $L_2(\mathbb{C}^n, \mathrm{dg}_n)$ of square-integrable functions on $\mathbb{C}^n$ with respect to the Gaussian measure

$$\mathrm{dg}_n(z) = \pi^{-n} e^{-|z|^2} \mathrm{d}\nu_n(z),$$

where $\mathrm{d}\nu_n(z) = \mathrm{d}x\mathrm{d}y$ is the standard Lebesgue plane measure on $\mathbb{C}^n = \mathbb{R}^{2n}$.

The Fock-Segal-Bargmann space

The Fock-Segal-Bargmann space $\mathcal{F}^2(\mathbb{C}^n)$ is the closure in $L_2(\mathbb{C}^n, \mathrm{dg}_n)$ of the set of all smooth functions satisfying the equations

$$\frac{\partial f}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) f = 0, \quad j = 1, 2, ..., n.$$

The Bargmann projection

There exists a unique orthogonal projection $\mathbf{P}$ from $L_2(\mathbb{C}^n, \mathrm{dg}_n)$ onto $\mathcal{F}^2(\mathbb{C}^n)$. This projection has the integral form

$$(\mathbf{P}f)(z) = \int_{\mathbb{C}^n} f(w)\overline{K_z(w)}\mathrm{dg}_n(w), \tag{1}$$

where the function $K_z\colon \mathbb{C}^n \to \mathbb{C}^n$ is the *reproducing kernel* at a point z, and it is given by the formula

$$K_z(w) = e^{\overline{z}\cdot w} \quad w \in \mathbb{C}^n. \tag{2}$$

Toeplitz operators

Given $\varphi \in L_\infty(\mathbb{C}^n)$, the *Toeplitz operator* T_φ with defining symbol φ acts on the Fock space $\mathcal{F}^2(\mathbb{C}^n)$ by the rule $T_\varphi f = \mathbf{P}(f\varphi)$, where $\mathbf{P}$ stays for the orthogonal projection from $L_2(\mathbb{C}^n, \mathrm{dg}_n)$ onto $\mathcal{F}^2(\mathbb{C}^n)$.

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The Toeplitz operator T_φ has the following integral representation

$$(T_\varphi f)(z) = \pi^{-n} \int_{\mathbb{C}^n} f(w) e^{z\cdot\overline{w}} e^{-|w|^2} \varphi(w) \mathrm{d}\nu_n(w), \quad z \in \mathbb{C}^n.$$

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J. Isralowitz, K. Zhu, *Toeplitz operators on Fock spaces*. Integr. Equ. Oper. Theory **66** (2010), 593–611. http://dx.doi.org/10.1007/s00020-010-1768-9

Toeplitz operators with measures as symbols

Isralowitz and Zhu introduced the Toeplitz operators T_μ acting on the Fock space $\mathcal{F}^2(\mathbb{C}^n)$ with Borel regular measures μ as symbols in a weak sense as follows

$$(T_\mu f)(z) = \pi^{-n} \int_{\mathbb{C}^n} e^{z\cdot\overline{w}} f(w) e^{-|w|^2} \mathrm{d}\mu(w), \quad z \in \mathbb{C}^n. \tag{1}$$

M condition

If μ is a Complex Borel regular measure that satisfy the condition (M), namely

$$\int_{\mathbb{C}^n} |K_z(w)|^2 e^{-|w|^2} \mathrm{d}|\mu|(w) < \infty, \tag{1}$$

then the operator T_μ is well-defined on the dense subset of all finite linear combinations of kernel function.

Fock-Carleson type measures

J. Isralowitz, K. Zhu, *Toeplitz operators on Fock spaces*. Integr. Equ. Oper. Theory **66** (2010), 593–611. http://dx.doi.org/10.1007/s00020-010-1768-9

Definition.

A complex valued measure μ is called a *Fock-Carleson* type measure for $\mathcal{F}^2(\mathbb{C}^n)$ if there exists a constant $\omega(\mu) > 0$ such that for every $f \in \mathcal{F}^2(\mathbb{C}^n)$

$$\int_{\mathbb{C}^n} |f(w)|^2 e^{-|w|^2} \mathrm{d}|\mu|(w) \leq \omega(\mu) \, \|f\|^2_{\mathcal{F}^2(\mathbb{C}^n)} \tag{2}$$

Proposition

Let μ be a complex measure satisfying the M condition. Then the following conditions are equivalent:

a) The Toeplitz operator T_μ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$.

b) The sesquilinear form

$$\mathrm{F}(f,g) = \int_{\mathbb{C}^n} f(z)\overline{g(z)}e^{-|z|^2}\mathrm{d}\mu(z) \tag{3}$$

is well-defined and bounded in $\mathcal{F}^2(\mathbb{C}^n)$.

c) $\widetilde{\mu}(z) = \pi^{-n}\int_{\mathbb{C}^n} e^{-|z-w|^2}\mathrm{d}\mu(w)$ is bounded on $\mathbb{C}^n$.

d) For any $\mathbf{r} > 0$, there exists $C > 0$ such that

$$|\mu|(B_{\mathbf{r}}(z)) < C, \quad \text{for all } z \in \mathbb{C}^n. \tag{4}$$

e) μ is a Fock-Carleson type measure.

For a Fock-Carleson type measure μ the following norms are equivalent:

1. $\|\mu\|_1 = \|T_\mu\|$.
2. $\|\mu\|_2 = \sup_{z \in \mathbb{C}^n} |\widetilde{\mu}(z)|$.
3. $\|\mu\|_3 = \sup_{z \in \mathbb{C}^n} |\mu|(B_{\mathbf{r}}(z))$, where $\mathbf{r}$ is any fixed positive radius.
4. $\|\mu\|_4 = \sup_{\substack{f \in \mathcal{F}^2(\mathbb{C}^n) \\ \|f\|_2 = 1}} \left\{ \int_{\mathbb{C}^n} |f(w)|^2 e^{-|w|^2} \mathrm{d}|\mu|(w) \right\}$.

Localization operators

Weyl operator

Let $h \in \mathbb{C}^n$. The *Weyl operator* $\mathcal{W}_h$ on $L_2(\mathbb{C}^n, \mathrm{dg}_n)$ is a weighted translation given by the rule

$$\mathcal{W}_h f(z) = e^{z\cdot\overline{h} - \frac{|h|^2}{2}} f(z-h), \quad z \in \mathbb{C}^n. \tag{3}$$

Proposition

Let $h \in \mathbb{C}^n$. The following statements hold:

(a) The Weyl operator $\mathcal{W}_h$ is unitary, with $\mathcal{W}_{-h} = \mathcal{W}_h^{-1}$.

(b) If M_φ be the multiplication operator by $\varphi \in L_\infty(\mathbb{C}^n)$, then

$$\mathcal{W}_h M_\varphi \mathcal{W}_{-h} f = M_{\varphi \circ \tau_h} f, \quad f \in \mathcal{F}^2(\mathbb{C}^n). \tag{3}$$

(c) . If $z \in \mathbb{C}^n$, then

$$\mathcal{W}_h K_z(w) = e^{-\overline{z} \cdot h - \frac{|h|^2}{2}} K_{z+h}(w), \quad w \in \mathbb{C}^n, \tag{4}$$

where $K_z(w) = e^{\overline{z} \cdot w}$.

(d) . If $\varphi \in L_\infty(\mathbb{C}^n)$, then

$$\mathcal{W}_h T_\varphi \mathcal{W}_{-h} = T_{\varphi \circ \tau_h}. \tag{5}$$

(e) $\mathcal{W}_z \mathcal{W}_h = e^{-\dfrac{i\,\mathrm{Im}(z \cdot \overline{h})}{2}} \mathcal{W}_{z+h}$ for all $z, h \in \mathbb{C}^n$. (**Weyl commutation relations**)

(f) $\widetilde{\mathcal{W}_z A \mathcal{W}_{-z}}(w) = \widetilde{A}(w - z)$, for all $A \in \mathcal{B}(\mathcal{F}^2(\mathbb{C}^n))$.

Localization operators

Definition.

Let $\varphi \in L_\infty(\mathbb{C}^n)$ and $f \in \mathcal{F}^2(\mathbb{C}^n)$. Then the linear operator $L_\varphi^{(f)}$ given by

$$\left\langle L_\varphi^{(f)} g, h \right\rangle = \pi^{-n} \int_{\mathbb{C}^n} \varphi(z) \langle g, \mathcal{W}_z f \rangle \langle \mathcal{W}_z f, h \rangle \, \mathrm{d}\nu_n(z) \tag{6}$$

is so-called *the Gabor-Daubichies localization operator* with window f and symbol φ.

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Note that

$$L_\varphi^{(f)} g = \pi^{-n} \int_{\mathbb{C}^n} \varphi(z) \langle g, \mathcal{W}_z f \rangle \, \mathcal{W}_z f \, \mathrm{d}\nu_n(z)$$

Note that

$$L_{\varphi}^{(f)} = \mathbf{W}_f^* M_\varphi \mathbf{W}_f$$

Window

Symbol

where

$$\mathbf{W}_f h(z) = \langle h, \mathcal{W}_z f \rangle \,, \qquad \textbf{(Wavelet type transformation)}$$

$$\mathbf{W}_f^* h(\zeta) = \int_{\mathbb{C}^n} h(z) \left(\mathcal{W}_z f\right)(\zeta) \, \mathrm{d}\nu_n(z),$$

Cordero, E., Tabacco, A. (2004). Localization Operators Via Time-Frequency Analysis. In: Ashino, R., Boggiatto, P., Wong, M.W. (eds) Advances in Pseudo-Differential Operators. Operator Theory: Advances and Applications, vol 155. Birkhäuser, Basel. https://doi.org/10.1007/978-3-0348-7840-1_8

Transformada rapida de Fourier

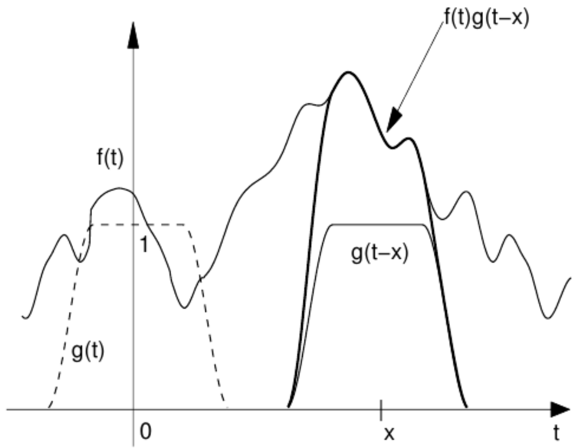


FIGURE 1. *The short-time Fourier transform.*

Haar system

In $L_2(\mathbb{R})$, the Haar function

$$\psi(x) := \begin{cases} 1, & \text{if } 0 \leq x < \frac{1}{2}; \\ -1, & \text{if } \frac{1}{2} \leq x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

may be used as a window function.

Image compression with the Haar Wavelet

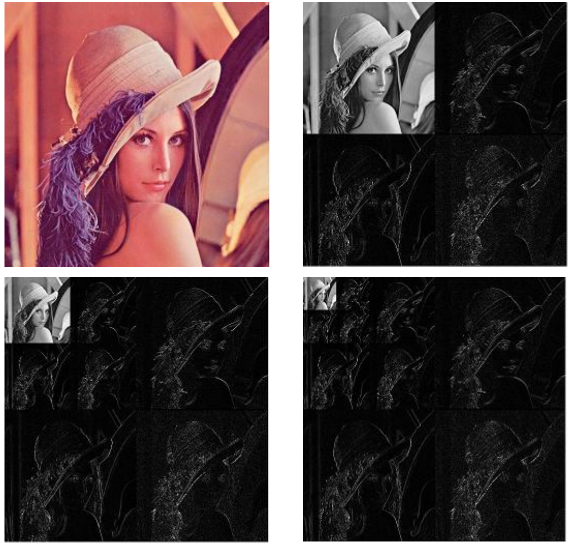


Figure 6.2: Comparison between JPEG and JPEG 2000. CR stands for compression ration and RMSE means root mean square error.

Fuente: *A Tutorial of the Wavelet Transform* Chun-Lin, Liu.

Example

For $f = \mathbf{1}$, we get for any $g, h \in \mathcal{F}^2(\mathbb{C}^n)$,

$$\left\langle L_\varphi^{(\mathbf{1})} g, h \right\rangle = (\pi)^{-n} \int_{\mathbb{C}^n} \varphi(z) \langle g, k_z \rangle \langle k_z, h \rangle \, \mathrm{d}\nu_n(z), \quad \text{by (3)} \quad \underbrace{\mathcal{W}_z \mathbf{1} = k_z}_{\text{normalized kernel}},$$

$$= \int_{\mathbb{C}^n} \varphi(z) g(z) \overline{h(z)} \, \mathrm{dg}_n(z), \quad \text{by reproducing property}$$

$$= \langle T_\varphi g, h \rangle, \quad g, h \in \mathcal{F}^2(\mathbb{C}^n).$$

Thus, $L_\varphi^{(\mathbf{1})} = T_\varphi$ for all $\varphi \in L_\infty(\mathbb{C}^n)$.

Example

if $e_j(z) = \dfrac{z^j}{\sqrt{j!}}$, where $j = (j_1, j_2, \ldots, j_n) \in \mathbb{Z}_+^n$ and $j! = j_1! j_2! \cdots j_n!$, then the family $\{e_j : j \in \mathbb{Z}_+^n\}$ is an orthonormal basis for $\mathcal{F}^2(\mathbb{C}^n)$. The localization opertor $L_\varphi^{(e_j)}$ is bounded for every $j \in \mathbb{Z}_+^n$. In particular, for $f(z) = \dfrac{e_1}{\sqrt{2}}$ and $f(z) = \dfrac{(e_1)^2}{2^{3/2}}$, respectively, we have

$$L_\varphi^{(f)} = T_{\varphi + 2\partial_1 \overline{\partial}_1 \varphi},$$

$$L_\varphi^{(f)} = T_{\varphi + 4\partial_1 \overline{\partial}_1 \varphi + 2\left(\partial_1 \overline{\partial}_1\right)^2 \varphi}, \qquad \text{Here } \partial_1 = \frac{\partial}{\partial z}, \quad \overline{\partial}_1 = \frac{\partial}{\partial \overline{z}}$$

for any φ which is either a polynomial in $z, \overline{z}$ or belongs to the algebra $B_c(\mathbb{C}^n)$ of Fourier-Stieltjes transforms of compactly supported complex measures on $\mathbb{C}^n$.

Proposition

For any $f \in \mathcal{F}^2(\mathbb{C}^n)$ and $\varphi \in L_\infty(\mathbb{C}^n)$, the localization operator $L_\varphi^{(f)}$ is bounded, and

$$\left\| L_\varphi^{(f)} \right\| \leq \|\varphi\|_\infty \|f\|^2.$$

M. Englis, Toeplitz operators and localization operators. Trans. Amer. Math. Soc. 361 (2009), 1039-1052. https://doi.org/10.1090/S0002-9947-08-04547-9

Proposición

For any polynomial $p \in \mathcal{F}^2(\mathbb{C}^n)$, there exists a constant coefficient linear partial differential operator $D = D(p)$ such that for any $\varphi \in BC^\infty(\mathbb{C}^n)$ (the space of all C^∞ functions on $\mathbb{C}^n$ whose partial derivatives of all orders are bounded),

$$L_\varphi^{(p)} = T_{D\varphi}, \qquad \text{on } \mathcal{F}^2(\mathbb{C}^n).$$

Explicitly, the operator D is given by

$$D(p) = \left[e^{\Delta/2}|p|^2\right]\Big|_{z\mapsto\overline{\partial},\overline{z}\mapsto 2\partial}$$

Here $e^{\Delta/2}$ should be understood as the infinite series

$$e^{\Delta/2} = \sum_{k=0}^{\infty} \frac{\Delta^k}{k!2^k}.$$

Localization operator with a trace-class operator as window

M. Englis, Toeplitz operators and localization operators. Trans. Amer. Math. Soc. 361 (2009), 1039-1052. https://doi.org/10.1090/S0002-9947-08-04547-9

Definition

let A be a bounded linear operator acting on $\mathcal{F}^2(\mathbb{C}^n)$. Then the " A-localization operator" with symbol φ and "window" A is the linear operator $L_\varphi^{(A)}$ with integral representation

$$L_\varphi^{(A)} = \pi^{-n} \int_{\mathbb{C}^n} \varphi(z) \mathcal{W}_z A \mathcal{W}_z^* \, \mathrm{d}\nu_n(z) \qquad \text{(weak sense).} \tag{7}$$

Remark

$$\begin{aligned} \left\langle L_\varphi^{(A)} f, g \right\rangle &= \pi^{-n} \int_{\mathbb{C}^n} \varphi(z) \left\langle A\mathcal{W}_z^* f, \mathcal{W}_z^* g \right\rangle \mathrm{d}\nu_n(z) \\ &= \pi^{-n} \int_{\mathbb{C}^n} \varphi(z) \left\langle A\mathcal{W}_{-z} f, \mathcal{W}_{-z} g \right\rangle \mathrm{d}\nu_n(z) \end{aligned}$$

Proposition

If A is trace-class, then the integral

$$L_\varphi^{(A)} = \pi^{-n} \int_{\mathbb{C}^n} \varphi(z) \mathcal{W}_z A \mathcal{W}_z^* \,\mathrm{d}\nu_n(z)$$

converges in the weak operator topology for any $\varphi \in L_\infty(\mathbb{C}^n)$, and

$$\left\| L_\varphi^{(A)} \right\|_{\mathrm{op}} \leq \|\varphi\|_\infty \|A\|_{\mathrm{tr}},$$

where $\|\cdot\|_{\mathrm{tr}}$ denotes the trace norm.

M. Englis, Toeplitz Operators and groups representations, Journal of Fourier Analysis and Applications 13, No. 3 (2007), 243-265.

Definition:(general case)

Let $A \in \mathcal{B}(\mathcal{F}^2(\mathbb{C}^n))$, G :=biholomorphic self-maps of $\mathbb{C}^n$ and H its corresponding Haar measure. Then for a function φ on G, the *A-Toeplitz operator with symbol* φ is given by

$$A_\varphi := \int_G \varphi(g) U_g^* A U_g \, \mathrm{d}\,\mathrm{H}(g)$$

whenever the integral exists (as usual, in the weak operator topology).

Localization with two admissible wavelets

$L_\varphi^{(A)}$ where the operator $A = \Phi \otimes \Psi$ and Φ, Ψ are two admissible wavelets.

$$\left\langle L_\varphi^{(\Phi\otimes\Psi)} f, g \right\rangle = \pi^{-n} \int_{\mathbb{C}^n} \varphi(z) \left\langle \mathcal{W}_{-z} f, \Psi \right\rangle \left\langle \Phi, \mathcal{W}_{-z} g \right\rangle \, \mathrm{d}\nu_n(z)$$

$A = \Phi \otimes \Phi$

$L_\varphi^{(A)}$ where the operator $A = \Phi \otimes \Phi$

$$\left\langle L_\varphi^{(\Phi\otimes\Phi)} f, g \right\rangle = \pi^{-n} \int_{\mathbb{C}^n} \varphi(z) \left\langle \mathcal{W}_{-z} f, \Phi \right\rangle \left\langle \Phi, \mathcal{W}_{-z} g \right\rangle \, \mathrm{d}\nu_n(z)$$
$$= \left\langle L_\varphi^{(\Phi)} f, g \right\rangle$$

In particular, $L_\varphi^{\mathbf{1}\otimes\mathbf{1}} = T_\varphi$ since

$$\mathcal{W}_z \left(\mathbf{1} \otimes \mathbf{1}\right) \mathcal{W}_z^* \cdot = \left\langle \mathcal{W}_z^* \cdot, \mathbf{1} \right\rangle \mathcal{W}_z \mathbf{1} = \left\langle \cdot, \mathcal{W}_z \mathbf{1} \right\rangle \mathcal{W}_z \mathbf{1} = k_z \otimes k_z.$$

In general, there is a way to relate a localization operator with a Toeplitz operator as follows.

Proposition

Let J be a finite set and $A = \sum_{j \in J} p_j \otimes q_j$, where p_j, q_j are polinomials in $\mathcal{F}^2(\mathbb{C}^n)$. Then there exists a unique linear partial differential operator $D = D^{(A)}$ (depending only on A) such that

$$L_\varphi^{(A)} = T_{D\varphi},$$

for every C^∞-function φ on $\mathbb{C}^n$ whose partial derivatives of all orders are bounded.

Let A be a trace-class operator. Then following the Zhu's idea to introduce Toeplitz operators with measures as symbols

in (7) we may modify a little the definition of the localization A-operator as follows:

$$L_{\varphi}^{(A)} f = \pi^{-n} \int_{\mathbb{C}^n} (\mathcal{W}_z A \mathcal{W}_z^* f) \overbrace{\varphi(z)\, \mathrm{d}\nu_n(z)}^{\mathrm{d}\mu(z)}$$

$$= \pi^{-n} \int_{\mathbb{C}^n} \mathcal{W}_z A \mathcal{W}_z^* f \, \mathrm{d}\mu(z).$$

Localization operator with a trace-class operator as window and a measure as symbol

Definition:

Let A be a trace-class operator and μ be a positive Borel regular measure. Then the localization A-operator relative to μ (Localization operator with window A and symbol μ), denoted by $L_{\mu}^{(A)}$, is the linear operator

$$L_{\mu}^{(A)} f = \pi^{-n} \int_{\mathbb{C}^n} \mathcal{W}_z A \mathcal{W}_z^* f \, \mathrm{d}\mu(z). \tag{7}$$

C. A. Berger and L. A. Coburn, Toeplitz operators on the Segal-Bargmann space, Trans. Amer. Math. Soc. 301 (1987), 813-829.
`https://doi.org/10.1090/S0002-9947-1987-0882716-4`

Example

If $\mathrm{d}\mu(z) = e^{-|z|^2}\mathrm{d}\nu_n(z) = \mathrm{d}g_n(z)$, then

$$\underbrace{L_\mu^{(A)} = \pi^{-n} \int_{\mathbb{C}^n} \mathcal{W}_z A \mathcal{W}_z^* \, \mathrm{d}g_n(z) = T_{\widetilde{A}}}_{\text{it is a classical Toeplitz operator}}$$

Remark

- Let A be a trace-class operator and μ be a positive Borel regular measure that satisfy the following M' condition:

$$\textbf{(M')} \qquad \int_{\mathbb{C}^n} |(z-w)^j|^2 e^{-|z-w|^2}\,\mathrm{d}|\mu|(w) < \infty \qquad \text{for all } z \in \mathbb{C}^n,$$

then $L_\mu^{(A)}$ is densely defined

- If A is a trace-class self-adjoint operator and μ is a positive Borel regular measure then $L_\mu^{(A)}$ is symmetric.

If μ satisfies the M' condition then

$$\widetilde{L_\mu^{(A)}}(z) = \pi^{-n} \int_{\mathbb{C}^n} \widetilde{A}(z-w)\mathrm{d}\mu(w). \tag{7}$$

It is easily seen in case of boundedness of $L_\mu^{(A)}$ that

$$\widetilde{L_\mu^{(A)}}(z) = \frac{\left\langle L_\mu^{(A)} K_z, K_z \right\rangle}{\langle K_z, K_z \rangle}, \quad z \in \mathbb{C}^n.$$

Main result

Proposition

Let A be a self-adjoint trace-class operator and μ be a positive Borel regular measure satisfying the M' condition. Then the following statements are equivalent:

1. The localization A-operator $L_\mu^{(A)}$ relative to μ is bounded
2. The sesquilinear form

$$\mathbf{F}(f,g) = \pi^{-n} \int_{\mathbb{C}^n} \langle A\mathcal{W}_{-z} f, \mathcal{W}_{-z} g \rangle \, \mathrm{d}\mu(z) = \left\langle L_\mu^{(A)} f, g \right\rangle$$

   is bounded
3. The Berezin transform $\widetilde{L_\mu^{(A)}}$ belongs to $L_\infty(\mathbb{C}^n)$
4. μ is a Fock-Carleson type measure.

Main result

Proof

3⇒4 Suppose that $\widetilde{L_\mu^{(A)}} \in L_\infty(\mathbb{C}^n)$. Since $A = \sum_{n\in\mathbb{N}} \lambda_n u_n \otimes u_n$ for some orthonormal basis $(u_n)_{n\in\mathbb{N}}$ for $\mathcal{F}^2(\mathbb{C}^n)$, we have

$$\widetilde{L_\mu^{(A)}}(w) = \sum_{n\in\mathbb{N}} \lambda_n e^{-|w|^2} |u_n(w)|^2 \geq 0, \quad w \in \mathbb{C}^n. \tag{8}$$

Futhermore, the Toeplitz operator $T_{\widetilde{L_\mu^{(A)}}}$ with symbol $\widetilde{L_\mu^{(A)}}$ is bounded.

Main result

J. Isralowitz, K. Zhu, Toeplitz operators on Fock spaces. Integr. Equ. Oper. Theory 66 (2010), 593–611. http://dx.doi.org/10.1007/s00020-010-1768-9

Proof

Now, by [1, Corollary 8], for any $\mathbf{r} > 0$ there exists $C > 0$ such that

$$\int_{B_{\mathbf{r}}(z)} \widetilde{L_{\mu}^{(A)}}(w) \mathrm{d}\nu_n(w) \leq C, \quad \text{for all } z \in \mathbb{C}^n.$$

Therefore, by Tonelli's theorem we get

$$C \geq \int_{B_{\mathbf{r}}(z)} \widetilde{L_{\mu}^{(A)}} \mathrm{d}\nu_n(w) \geq \pi^{-n} \int_{B_{\mathbf{r}}(z)} \int_{B_{2\mathbf{r}}(0)} \widetilde{A}(\omega) \mathrm{d}\nu_n(\omega) \mathrm{d}\mu(\zeta) = \pi^{-n} C_r^A \, \mu(B_{\mathbf{r}}(z)),$$

Main result

Proof.

where

$$C_{\mathbf{r}}^{A} = \int_{B_{2\mathbf{r}}(0)} \widetilde{A}(\omega)\mathrm{d}\nu_n(\omega) = \sum_{n\in\mathbb{N}} \lambda_n \int_{B_{2\mathbf{r}}(0)} e^{-|w|^2} \left|u_n(w)\right|^2 \mathrm{d}\nu_n(w) \neq 0.$$

Thus, μ is a Fock-Carleson type measure.

Main result

Proof

4⇒1 Suppose that μ is a positive Fock-Carleson type measure on $\mathcal{F}^2(\mathbb{C}^n)$ and A is self-adjoint trace-class operator. Since $L_\mu^{(A)}$ is densely defined, then it is bounded by the Hellinger-Toeplitz theorem because $L_\mu^{(A)}$ is symmetric.

Main result

Remark

If A is not self-adjoint, then $A = A_{Re} + iA_{Im}$, where A_{Re} and A_{Im} are trace-class and self-adjoint, $L_\mu^{(A)} = L_\mu^{(A_{Re})} + iL_\mu^{(A_{Im})}$, furthermore, $L_\mu^{(A_{Re})}$, and $L_\mu^{(A_{Im})}$ are symmetric. Now, we apply the theorem to $L_\mu^{(A_{Re})}$ and $L_\mu^{(A_{Im})}$. Thus, we have that

$$\left|L_\mu^{(A)} f(z)\right|^2 \leq 2 \left(\max\left\{\left\|L_\mu^{(A_{Re})}\right\|_{\text{op}}, \left\|L_\mu^{(A_{Im})}\right\|_{\text{op}}\right\}\right)^2 \|f\|^2.$$

Therefore, $\|L_\mu^{(A)}\|_{\text{op}} \leq \sqrt{2} \max\left\{\left\|L_\mu^{(A_{Re})}\right\|_{\text{op}}, \left\|L_\mu^{(A_{Im})}\right\|_{\text{op}}\right\}$.

C. A. Berger and L. A. Coburn, Toeplitz operators on the Segal-Bargmann space, Trans. Amer. Math. Soc. 301 (1987), 813-829. https://doi.org/10.1090/S0002-9947-1987-0882716-4

Remark

If μ is a Fock-Carleson type measure for $\mathcal{F}^2(\mathbb{C}^n)$, then $\widetilde{\mu} \in L_\infty(\mathbb{C}^n)$ and hence

$$L^{(A)}_{\widetilde{\mu}} f = \pi^{-n} \int_{\mathbb{C}^n} \widetilde{\mu}(z) \mathcal{W}_z A \mathcal{W}_z^* f \, \mathrm{d}\nu_n(z)$$

is bounded. Now, by the Fubini's theorem and the Weyl commutation relations we have

$$\begin{aligned} L^{(A)}_{\widetilde{\mu}} f &= \pi^{-n} \int_{\mathbb{C}^n} \mathcal{W}_\zeta L^{(A)}_{\mu} \mathcal{W}_\zeta^* f \, \mathrm{d}g_n(\zeta), \\ &= T_{\widetilde{L^{(A)}_{\mu}}} f, \quad f \in \mathcal{F}^2(\mathbb{C}^n). \end{aligned}$$

A-Toeplitz type operators on $\mathcal{F}^2(\mathbb{C}^n)$ with measures as symbols

A-localization operators

Let A be a trace-class operator and μ be a positive Borel regular measure that satisfy the M' condition. Then the localization A-operator relative to μ, denoted by $L_\mu^{(A)}$, is the linear operator

$$L_\mu^{(A)} f = \pi^{-n} \int_{\mathbb{C}^n} \mathcal{W}_z A \mathcal{W}_z^* f \, \mathrm{d}\mu(z).$$

In particular,

If $A = E_j = e_j \otimes e_j$, $j \in \mathbb{Z}_+^n$, then

$$L_\mu^{(E_j)} f = \pi^{-n} \int_{\mathbb{C}^n} \mathcal{W}_z E_j \mathcal{W}_z^* f \, \mathrm{d}\mu(z) = \pi^{-n} \int_{\mathbb{C}^n} \langle f, \mathcal{W}_z e_j \rangle \, \mathcal{W}_z e_j \, \mathrm{d}\mu(z).$$

Here the family $\{e_j : j \in \mathbb{Z}_+^n\}$ is the orthonormal basis for $\mathcal{F}^2(\mathbb{C}^n)$ with $e_j(z) = \dfrac{z^j}{\sqrt{j!}}$, $j = (j_1, j_2, \ldots, j_n) \in \mathbb{Z}_+^n$ and $j! = j_1! j_2! \cdots j_n!$.

D. Suárez, A Generalization of Toeplitz operators on the Bergman space. Journal of Operator Theory 73, no. 2 (2015): 315–32.
http://www.jstor.org/stable/24718127.

A-Toeplitz operators with measures as symbols

Let $j \in \mathbb{Z}_+^n$. For a positive Borel regular measure μ satisfying the M' condition. Then the Toeplitz A-operator relative to μ, denoted by $\mathbf{T}_\mu^{(j)}$, is the linear operator

$$\mathbf{T}_\mu^{(j)} f = L_\mu^{(E_j)} f = \pi^{-n} \int_{\mathbb{C}^n} \mathcal{W}_z E_j \mathcal{W}_z^* f \, \mathrm{d}\mu(z). \qquad (8)$$

That is, $\mathrm{T}_{\mu}^{(j)}$ is the linear operator such that

$$\left\langle \mathbf{T}_{\mu}^{(j)} f, g \right\rangle = \pi^{-n} \int_{\mathbb{C}^n} \langle f, \mathcal{W}_z e_j \rangle \langle \mathcal{W}_z e_j, g \rangle \, \mathrm{d}\mu(z).$$

Properties

- $\left\langle \mathbf{T}_\mu^{(0)} f, g \right\rangle = \pi^{-n} \int_{\mathbb{C}^n} \langle f, \mathcal{W}_z \mathbf{1} \rangle \langle \mathcal{W}_z \mathbf{1}, g \rangle \, \mathrm{d}\mu(z) = \langle T_\mu f, g \rangle$.
- For a positive Borel regular measure μ satisfying the M' condition, then it is defined a function $\widetilde{\mathbf{T}_\mu^{(j)}}$ on $\mathbb{C}^n$ as follows:

$$\begin{aligned} \widetilde{\mathbf{T}_\mu^{(j)}}(z) &= \pi^{-n} \int_{\mathbb{C}^n} \widetilde{E_j}(z-w) \, \mathrm{d}\mu(w), \quad \text{where } \widetilde{E_j}(w) = |w^j|^2 \, e^{-|w|^2} \\ &= \pi^{-n} \int_{\mathbb{C}^n} |(z-w)^j|^2 \, e^{-|z-w|^2} \mathrm{d}\mu(w). \end{aligned} \tag{8}$$

In case of boundedness of $\mathbf{T}_\mu^{(j)}$ that

$$\widetilde{\mathbf{T}_\mu^{(j)}}(z) = \frac{\left\langle \mathbf{T}_\mu^{(j)} K_z, K_z \right\rangle}{\langle K_z, K_z \rangle}, \quad z \in \mathbb{C}^n.$$

Theorem

Let $\mu \in \mathfrak{B}_{reg,+}(\mathbb{C}^n)$ and $j \in \mathbb{Z}_+^n$. Then the following statements are equivalent:

1. $\mathbf{T}_\mu^{(j)}$ is bounded in $\mathcal{F}^2(\mathbb{C}^n)$
2. The sesquilinear form $\mathbf{F}_j \colon \mathcal{F}^2(\mathbb{C}^n) \times \mathcal{F}^2(\mathbb{C}^n) \to \mathbb{C}$ given by

$$\mathbf{F}_j(f,g) = \pi^{-n} \int_{\mathbb{C}^n} \langle f, \mathcal{W}_z e_j \rangle \langle \mathcal{W}_z e_j, g \rangle \, \mathrm{d}\mu(z)$$

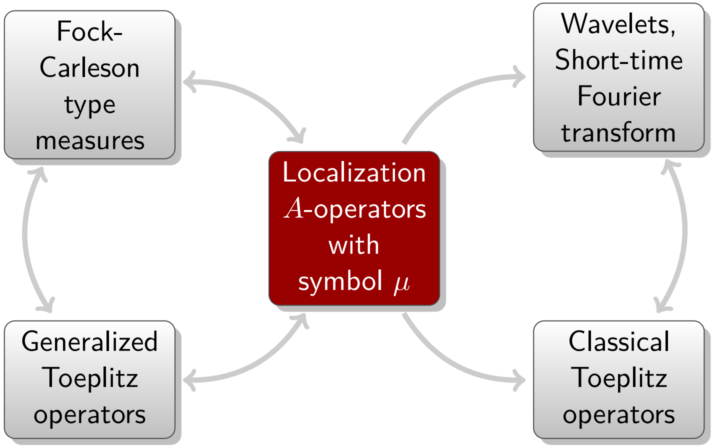
   is bounded
3. The Berezin transform $\widetilde{\mathbf{T}_\mu^j}$ of $\mathbf{T}_\mu^j$ belongs to $L_\infty(\mathbb{C}^n)$
4. μ is a Fock-Carleson type measure

Open problems

- ¿What can we say about the algebra $\mathcal{L}$ generated by $\{L_\mu^{(A)} : A \in S_1 \text{ and } \mu \text{ FC-type measure}\}$?
- Note that $\mathrm{B}\, T_\varphi\, \mathrm{B}^*$ is a pseudo-differential operator for any Toeplitz operator T_φ, here B denotes the Bargmann transform from $\mathcal{F}^2(\mathbb{C}^n)$ onto $L_2(\mathbb{R}^n)$. ¿What can we say about $\mathrm{B}\, L_\mu^{(A)}\, \mathrm{B}^*$?
- ¿Is it possible to introduce $L_\mu^{(A)}$ for any bounded operator A?
- If $A \in S_1$ is injective (surjective), ¿What can we say about $L_\mu^{(A)}$?
- If $(\eta_z)_{z\in\mathbb{C}^n}$ is a coherent state system in $\mathcal{F}^2(\mathbb{C}^n)$ and $(Af)(z) = \langle f, \eta_z \rangle\, \eta_z$, ¿What can we say about $L_\mu^{(A)}$?
- If μ is a Fock-Carleson type measure for derivatives of order $k \in \mathbb{Z}_+^n$, i.e.,

$$\int_{\mathbb{C}^n} \left| \frac{\partial^k f(z)}{\partial z^k} \right|^2 e^{-|z|^2}\, \mathrm{d}\mu(z) \leq \omega_k(\mu) \|f\|^2_{\mathcal{F}^2(\mathbb{C}^n)},$$

  ¿What can we say about $L_\mu^{(A)}$?


Fock-Carleson type measures
Wavelets, Short-time Fourier transform
Localization A-operators with symbol μ
Generalized Toeplitz operators
Classical Toeplitz operators

Thank you!