Fresh look at polyanalytic type spaces

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Polyanalytic functions from the abstract point of view

Polyanalytic functions

We will deal with the functions defined in the domain D being either he unit disk $\mathbb{D}$ in the complex plane, or the entire plane $\mathbb{C}$. Recall that an n-differentiable function φ is called n-polyanalytic if it satisfies in D the equation

$$\frac{\partial^n \varphi}{\partial \overline{z}^n} = 0, \qquad n \in \mathbb{N}.$$

As known (Balk), the above condition is equivalent to the following its representations

$$\varphi = \sum_{q=0}^{n-1} \overline{z}^q f_q,$$

where all f_q are analytic functions.

Step to a general situation

Observe: two operators $\overline{z}I$ and $\frac{\partial}{\partial \overline{z}}$ act invariantly on the linear space of all smooth function in D, and satisfy the relation

$$\left[\frac{\partial}{\partial \overline{z}}, \overline{z}\right] = \frac{\partial}{\partial \overline{z}}\overline{z} - \overline{z}\frac{\partial}{\partial \overline{z}} = I.$$

This is a particular case of the following general situation.

Lemma

Let $\mathfrak{L}$ be a linear space, which is invariant under the action of the two operators $\mathfrak{a}$ and $\mathfrak{b}$ and satisfying therein the relation

$$[\mathfrak{a}, \mathfrak{b}] = I.$$

Then an element $h \in \mathfrak{L}$ satisfies the equation $\mathfrak{a}^n h = 0$ if and only if it admits the representation

$$h = \sum_{s=0}^{n-1} \mathfrak{b}^s h_s, \qquad \text{with} \quad h_s \in \ker \mathfrak{a}.$$

Observation

Let us mention some properties.

As known $[\mathfrak{a}, \mathfrak{b}] = I$ easily implies $[\mathfrak{a}, \mathfrak{b}^n] = n\mathfrak{b}^{n-1}$, for all $n \in \mathbb{N}$, which, in turn, yields that for each $h \in \ker \mathfrak{a}$ and all $n \in \mathbb{N}$,

$$\mathfrak{a}\mathfrak{b}^n h = n\mathfrak{b}^{n-1} h.$$

By induction, for all $h \in \ker \mathfrak{a}$, we have then

$$\mathfrak{a}^k \mathfrak{b}^n h = \begin{cases} n(n-1)\cdots(n-k+1)\mathfrak{b}^{n-k} h, & \text{if} \quad k < n \\ n! h, & \text{if} \quad k = n \\ 0, & \text{if} \quad k > n \end{cases}.$$

Several important properties

Introduce the following subspaces of $\mathfrak{L}$

$$L_{[1]} = \ker \mathfrak{a} = \{h \in \mathcal{L} : \ \mathfrak{a}h = 0\} \quad \text{and} \quad L_{[n]} := \mathfrak{b}^{n-1} L_{[1]}, \quad n \in \mathbb{N},$$

Denote then by $\mathcal{L}_0$ the subspace of $\mathcal{L}$ formed by finite linear combinations of elements from the linear subsets $L_{[n]}$, $n \in \mathbb{N}$. Unless otherwise specified, in what follows we will always consider operators $\mathfrak{a}$ and $\mathfrak{b}$ (and those generated by them) in their action on elements of $\mathcal{L}_0$.

Lemma

The kernel of the operator $\mathfrak{b}$ is trivial, $\ker \mathfrak{b}|_{\mathcal{L}_0} = \{0\}$.

Corollary

For each $n \in \mathbb{N}$, the operator $\mathfrak{b}$ maps linearly independent elements of $L_{[n]}$ onto linearly independent elements of $L_{[n+1]}$, implying thus that $\dim L_{[n]} = \dim L_{[n+1]}$.

Several important properties

Corollary

For all $k \neq n$, the intersection of the subspaces $L_{[k]}$ and $L_{[n]}$ is trivial, $L_{[k]} \cap L_{[n]} = \{0\}$.

Note that the above Lemma admits the following equivalent reformulation.

Lemma

An element $h \in \mathfrak{L}$ satisfies the equation $\mathfrak{a}^n h = 0$ if and only if

$$h \in L_{[1]} + L_{[2]} + \ldots + L_{[n]}.$$

Extended Fock space construction

Fock space formalism

We add now a Hilbert space structure to the above pure linear algebra considerations.
A formal construction of the Fock space is based on the two formally adjoint operators $\mathfrak{a}$ and $\mathfrak{b} = \mathfrak{a}^\dagger$ defined on a dense linear subset of a separable Hilbert space $\mathcal{H}$, where they act invariantly and satisfy the commutation relation $[\mathfrak{a}, \mathfrak{a}^\dagger] = I$.
There also exists a normalized element $|0\rangle := \Phi_0 \in \mathcal{H}$, $\|\Phi_0\| = 1$, called the vacuum vector, such that $\mathfrak{a}\Phi_0 = 0$, and the linear span of elements $(\mathfrak{a}^\dagger)^n \Phi_0$ with $n \in \mathbb{Z}_+$ is dense in $\mathcal{H}$.

Example (Berezin)

Let $\mathcal{H} = F_2(\mathbb{C})$ be the space of all anti-analytic functions $f(\overline{z})$ endowed with the scalar product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{C}} f(\overline{z}) \overline{g(\overline{z})} e^{-|z|^2} dxdy, \quad \overline{z} = x - iy,$$

and $\mathfrak{a} = \frac{\partial}{\partial \overline{z}}$, $\mathfrak{a}^\dagger = \overline{z}$ with $\Phi_0 = 1$.

Extended Fock space construction

Introduce the three-dimensional Heisenberg algebra $\mathbb{H}_3 = \{\mathfrak{a}, \mathfrak{b}, 1\}$ with commutators $[\mathfrak{a}, \mathfrak{b}] = 1$ and $[\mathfrak{a}, 1] = [\mathfrak{b}, 1] = 0$.
Extended Fock space construction is given by the representation of the algebra $\mathbb{H}_3$ in a separable Hilbert space $\mathcal{H}$:

There are a separable Hilbert space $\mathcal{H}$ and operators $\mathfrak{a}$ and $\mathfrak{b}$ defined on their natural (maximal) domains $\mathcal{D}_\mathfrak{a}$ and $\mathcal{D}_\mathfrak{b}$ both dense in $\mathcal{H}$ and such that
- there is a linear subset $\mathcal{D} \subset \mathcal{D}_\mathfrak{a} \cap \mathcal{D}_\mathfrak{b}$, dense in $\mathcal{H}$ and invariant under the action of $\mathfrak{a}$ and $\mathfrak{b}$, on which they satisfy the relation

$$[\mathfrak{a}, \mathfrak{b}] = I;$$

- the linear subset $\ker \mathfrak{a}|_\mathcal{D} := L_{[1]} = \{h \in \mathcal{D} : \ \mathfrak{a}h = 0\}$ spanned by vacuum vectors, is non-trivial with $\dim L_{[1]} \geq 1$;
- the set $\mathcal{D}_0$, formed by finite linear combinations of elements from the linear subsets $L_{[n]} := \mathfrak{b}^{n-1} L_{[1]}$, $n \in \mathbb{N}$, is dense in $\mathcal{H}$.

Remarks

The classical Fock formalism corresponds to the case when $\mathfrak{b} = \mathfrak{a}^\dagger$ and $L_{[1]}$ is one-dimensional, being generated by a single element $|0\rangle := \Phi_0 \in \mathcal{H}$.

The operators $\mathfrak{a}$ and $\mathfrak{b}$ generate in fact the representation of the universal enveloping algebra of $\mathbb{H}_3$ in $\mathcal{D}_0 \subset \mathcal{H}$.

As examples show, the subspases $L_{[n]}$ may or may not be closed, as well as, they may or may not be mutually orthogonal to each other.

Examples

Two classical examples: $\mathcal{H} = L_2(\mathbb{D}, d\nu_\lambda)$, where $\lambda > -1$ and the measure $d\nu_\lambda$ is given by

$$d\nu_\lambda(z) = (\lambda + 1)(1 - |z|^2)^\lambda dA(z), \quad dA(z) = \frac{1}{\pi} dxdy,$$

and $\mathcal{H} = L_2(\mathbb{C}, d\mu_\alpha)$, where the Gaussian measure $d\mu_\alpha$ is given by

$$d\mu_\alpha(z) = \alpha e^{-\alpha|z|^2} dA(z), \qquad \text{for} \quad \alpha > 0.$$

In both cases, $\mathfrak{a} = \frac{\partial}{\partial \overline{z}}$ and $\mathfrak{b} = \overline{z}$ is the multiplication by $\overline{z}$ operator; $\mathcal{D} = \mathcal{D}_0$ consists of finite linear combinations of the monomials $m_{p,q} = z^p \overline{z}^q$, $p, q \in \mathbb{Z}_+$, and $\ker \mathfrak{a}$ coincides with the set of all analytic polynomials (polynomials on z).
Note that the linear set $\mathcal{D} = \mathcal{D}_0$ is nothing but the set of all polyanalytic polynomials, i.e., the set of all functions

$$\varphi = \sum_{q=0}^{n-1} \overline{z}^q f_q,$$

where all f_q are analytic polynomials.

Examples

A natural extension of the previous example is as follows:

Let D be either $\mathbb{D}$ or $\mathbb{C}$. We set $\mathcal{H}$ for any weighted Hilbert space $L_2(D, \omega)$, with the probability measure $d\nu(z) = \omega(|z|)dA(z)$, whose radial weight function $\omega : D \to \mathbb{R}_+$ is such that the linear span of the monomials $m_{p,q} := z^p \overline{z}^q$, for all $p,\ q \in \mathbb{Z}_+$, is dense in $\mathcal{H}$.
Here, as in the previous example, $\mathfrak{a} = \frac{\partial}{\partial \overline{z}}$ and $\mathfrak{b} = \overline{z}$, $\mathcal{D} = \mathcal{D}_0$ is the linear set of all polyanalytic polynomials, and $\ker \mathfrak{a}$ coincides with the set of all analytic polynomials.

In the above examples $\ker \mathfrak{a}$ is infinite dimensional, and the operators $\mathfrak{a}$ and $\mathfrak{b}$ are not formally adjoint. Indeed,

$$\langle \mathfrak{a} z^n, z^{n+1} \rangle = \langle 0, z^{n+1} \rangle = 0,$$

while

$$\langle z^n, \mathfrak{b} z^{n+1} \rangle = \langle z^n, \overline{z} z^{n+1} \rangle = \langle z^{n+1}, z^{n+1} \rangle = \|z^{n+1}\|^2 \neq 0.$$

Examples

Let $\mathcal{H} = L_2(\mathbb{C}, d\mu)$ be the Hilbert space of square-integrable functions on $\mathbb{C}$ with the Gaussian measure

$$d\mu(z) = e^{-z\cdot\bar{z}} dA(z) \ .$$

Introduce

$$\mathfrak{a} = \frac{\partial}{\partial \overline{z}} - \omega z \qquad \text{and} \qquad \mathfrak{b} = \overline{z}.$$

The functions φ, satisfying the equation $\mathfrak{a}\varphi = 0$, are the particular case of the so-called generalized analytic functions introduced and studied by I. Vekua. Generically they are the functions that satisfy the equation

$$\frac{\partial \varphi}{\partial \overline{z}} + A(z)\varphi + B(z)\overline{\varphi} = 0.$$

In our case $A(z) = -\omega z, \ \ B(z) = 0$.

It is easy to see that $[\mathfrak{a}, \mathfrak{b}] = I$.

Examples

Of course we can also consider:

- $\mathcal{H} = L_2(\mathbb{D}, d\nu_\lambda)$ and $\mathfrak{a} = \frac{\partial}{\partial z}$, $\mathfrak{b} = z$;

- $\mathcal{H} = L_2(\mathbb{C}, d\mu)$ and $\mathfrak{a} = \frac{\partial}{\partial z}$, $\mathfrak{b} = z$;

- $\mathcal{H} = L_2(\mathbb{C}, d\mu)$ and $\mathfrak{a} = \frac{\partial}{\partial z} - \omega\bar{z}$ $\mathfrak{b} = z$.

These examples leed to the anty-polyanalytic type spaces.

Examples

We give also two examples with $\mathfrak{b} = \mathfrak{a}^\dagger$.

Take $\mathcal{H} = L_2(\mathbb{C}, d\mu)$ with the Gaussian measure

$$d\mu(z) = e^{-z\cdot\bar{z}} dA(z) \ ,$$

And let

$$\mathfrak{a} = \frac{\partial}{\partial \bar{z}} \qquad \text{and} \qquad \mathfrak{a}^\dagger = -\frac{\partial}{\partial z} + \bar{z}.$$

Again, let $\mathcal{D} = \mathcal{D}_0$ be the linear set of all polyanalytic polynomials, and $\ker \mathfrak{a}$ coincides with the set of all analytic polynomials.

Note that the closure in $L_2(\mathbb{C}, d\mu)$ of the $\ker \mathfrak{a}$ coincides with the classical Fock space $F^2(\mathbb{C})$, being the closed subspace of $L_2(\mathbb{C}, d\mu)$, which consists of all analytic in $\mathbb{C}$ functions.

Examples

Again let $\mathcal{H} = L_2(\mathbb{C}, d\mu)$ with the Gaussian measure

$$d\mu(z) = e^{-z\cdot\bar{z}} dA(z) \ .$$

Then, for $\omega < \frac{1}{2}$. consider

$$\begin{aligned} \mathfrak{a}_\omega &= \frac{1}{\sqrt{1-2\omega}} \left(\frac{\partial}{\partial \bar{z}} - \omega z \right) \\ \mathfrak{a}_\omega^\dagger &= \frac{1}{\sqrt{1-2\omega}} \left(-\frac{\partial}{\partial z} + (1-\omega)\bar{z} \right) \end{aligned}$$

In this case the closure in $L_2(\mathbb{C}, d\mu)$ of the $\ker \mathfrak{a}$ coincides with the Vekua space $\mathcal{V}^2(\omega, \mathbb{C})$, to be described later on.

For $\omega = 0$, we return to the previous example.

Properties, continuation

Having at hand the Hilbert space structure, we extend now the properties of the spaces $L_{[n]}$, setting $\mathfrak{L}_0 = \mathcal{D}_0$.
Let us order the system of subspaces $\{L_{[n]}\}_{n\in\mathbb{Z}_+}$ by the rule

$$L_{[n_1]} \prec L_{[n_2]} \qquad \text{if only if} \qquad n_1 < n_2.$$

The following lemma justifies the names lowering and raising for the operators $\mathfrak{a}$ and $\mathfrak{b}$, respectively.

Lemma

For each $n \in \mathbb{Z}_+$, the operators $\mathfrak{a}$ and $\mathfrak{b}$, restricted correspondingly on $L_{[n+1]}$ and $L_{[n]}$, act as isomorphisms between the following spaces

$$\mathfrak{a}|_{L_{[n+1]}} : L_{[n+1]} \longrightarrow L_{[n]} \quad \text{and} \quad \mathfrak{b}|_{L_{[n]}} : L_{[n]} \longrightarrow L_{[n+1]}.$$

Moreover, the operators $\mathfrak{ab}$ and $\mathfrak{ba}$, being restricted on $L_{[n]}$ and $L_{[n+1]}$, respectively, act as the scalar operators,

$$\mathfrak{ab}|_{L_{[n]}} = nI : L_{[n]} \longrightarrow L_{[n]} \quad \text{and} \quad \mathfrak{ba}|_{L_{[n+1]}} = nI : L_{[n+1]} \longrightarrow L_{[n+1]}.$$

Properties, continuation

For each $n \in \mathbb{N}$, the operator $\mathfrak{ab}$ can be extended by continuity from $L_{[n]}$ to its closure $\overline{L_{[n]}} = \operatorname{clos} L_{[n]}$, so that $\mathfrak{ab}|_{\overline{L_{[n]}}} = nI$.
The operator $\mathfrak{ab}$, initially defined on $\mathcal{D}_0$, can be extended to a wider domain being the linear span of $\overline{L_{[n]}}$, and even more to $\mathcal{D}^{\#} = \left\{ f = \sum_{n\in\mathbb{N}} f_n \in \mathcal{H} : \ f_n \in \overline{L_{[n]}} \text{ and } \sum_{n\in\mathbb{N}} n f_n \in \mathcal{H} \right\}$.

Corollary

Each space $\overline{L_{[n]}}$ is the invariant subspace of the operator $\mathfrak{ab}$, defined on $\mathcal{D}^{\#}$. All of them are eigenspaces of $\mathfrak{ab}$, whose corresponding eigenvalues are n.

Corollary

For all $k \neq n$, the intersection of the closed subspaces $\overline{L_{[k]}}$ and $\overline{L_{[n]}}$ is trivial, $\overline{L_{[k]}} \cap \overline{L_{[n]}} = \{0\}$.

Corollary

Any finite number of spaces $\overline{L_{[k_1]}}$, $\overline{L_{[k_2]}}$, ..., $\overline{L_{[k_n]}}$ are linearly independent.

Direct and orthogonal sum

The linear span of finite number H_1, H_2, ...,H_n of linearly independent subspaces of $\mathcal{H}$ is called the direct sum and is denoted by

$$H_1 \dot{+} H_2 \dot{+} \ldots \dot{+} H_n,$$

in case when they are additionally pairwise orthogonal, we write

$$H_1 \oplus H_2 \oplus \ldots \oplus H_n,$$

and call it the orthogonal sum.

Note that, given even just two closed subspaces H_1 and H_2 with trivial intersection, $H_1 \cap H_2 = \{0\}$, (equivalently being linearly independent), their direct sum $H_1 \dot{+} H_2$ may not be closed.

It depends on the so-called minimal angle between them.

Minimal angle between subspaces

The minimal angle $\varphi^{(m)}(H_1, H_2)$ between two closed subspaces H_1 and H_2 of a Hilbert space H is defined as

$$\cos\varphi^{(m)}(H_1, H_2) = \sup\{|\langle x, y\rangle| : x \in H_1,\ y \in H_2 \text{ and } \|x\| = \|y\| = 1\}.$$

Recall in this connection the criterion of when the direct sum of two closed spaces is closed.

Lemma (I. Gohberg, A. Markus, '59)

The direct sum of two closed subspaces, that intersect only by zero, is closed if and only if the minimal angle between them is grater then zero.

For each $n \in \mathbb{N}$, the space $L_{[1]} \dot{+} L_{[2]} \dot{+} \ldots \dot{+} L_{[n]}$ is not necessarily closed, even in the case when all $L_{[k]}$ are closed.

An example will be given later on.

We define thus

$$L_n = \mathrm{clos}\,(L_{[1]} \dot{+} L_{[2]} \dot{+} \ldots \dot{+} L_{[n]}) \;=\; \mathrm{clos}\,(\overline{L_{[1]}} \dot{+} \overline{L_{[2]}} \dot{+} \ldots \dot{+} \overline{L_{[n]}}),$$

with the convention that $L_1 = \overline{L_{[1]}}$.
There might be elements $h \in L_n$ with $\mathfrak{a}^n h = 0$ such that

$$h = \sum_{k=0}^{n-1} \overline{z}^k h_k, \quad \text{but} \quad \overline{z}^k h_k \notin L_{[k]}.$$

Note that the nested spaces L_n form an infinite flag

$$L_1 \;\subset\; L_2 \;\subset\; \ldots \;\subset\; L_k \;\ldots\; \subset \bigcup_{n\in\mathbb{N}} L_n \;\subset\; \mathcal{H},$$

and the density of $\mathcal{D}_0$ in $\mathcal{H}$ implies

$$\mathcal{H} = \mathrm{clos}\left(\bigcup_{n\in\mathbb{N}} L_n\right),$$

Introduce also the spaces $L_{(n)} = L_n \ominus L_{n-1} = L_n \cap L_{n-1}^{\perp}$, then the equivalent representation of $\mathcal{H}$ is

$$\mathcal{H} = \bigoplus_{n\in\mathbb{N}} L_{(n)}.$$

Lie-algebraic characterization

Consider three invariantly acting on $\mathcal{D}_0$ operators

$$\begin{aligned} J_n^+ &= \mathfrak{b}^2\mathfrak{a} - (n-1)\,\mathfrak{b}, \\ J_n^0 &= \mathfrak{b}\mathfrak{a} - \tfrac{n-1}{2}, \\ J_n^- &= \mathfrak{a}. \end{aligned}$$

For all $n \in \mathbb{C}$ these operators obey the $\mathfrak{sl}(2)$-algebra commutation relations $[J_n^-, J_n^+] = 2J_n^0$ and $[J_n^\pm, J_n^0] = \mp J_n^\pm$.
For $n \in \mathbb{N}$ the space

$$L_{[1]} \dot{+} L_{[2]} \dot{+} \ldots \dot{+} L_{[n]}$$

is the maximal by inclusion subspace in $\mathcal{D}_0$, which is invariant under the action of the operators J_n^+, J_n^0 and J_n^-.

The space L_n thus can be defined alternatively as the closure of the maximal by inclusion subspace in $\mathcal{D}_0$, which is invariant for the action the operators J_n^+, J_n^0, J_n^- in $\mathcal{H}$, obeying the $\mathfrak{sl}(2)$-algebra commutation relations.

Additional properties in case of $\mathfrak{b} = \mathfrak{a}^\dagger$

Proposition

Different subspaces $L_{[n]}$ and $L_{[m]}$ are orthogonal to each other. For all $k = 2, 3, \ldots$, the raising operator

$$\frac{1}{\sqrt{k-1}}\, \mathfrak{a}^\dagger|_{\overline{L}_{[k-1]}} : \overline{L}_{[k-1]} \longrightarrow \overline{L}_{[k]}$$

is an isometric isomorphism, and the lowering operator

$$\frac{1}{\sqrt{k-1}}\, \mathfrak{a}|_{\overline{L}_{[k]}} : \overline{L}_{[k]} \longrightarrow \overline{L}_{[k-1]},$$

is its inverse.

The proposition implies that the domain $\mathcal{D}_0$ of the operators $\mathfrak{a}$ and $\mathfrak{a}^\dagger$ can be extended to a wider domain $\mathcal{D}_{ext}$ being the linear span of all $\overline{L}_{[k]}$, and that $\ker \mathfrak{a}|_{\mathcal{D}_{ext}} = \overline{L}_{[1]}$ is the closed subspace of $\mathcal{H}$.

The mutual orthogonality of the spaces $\overline{L}_{[k]}$ implies that

$$L_n = \mathrm{clos}\left(L_{[1]} \dot{+} L_{[2]} \dot{+} \ldots \dot{+} L_{[n]}\right) = \overline{L_{[1]}} \oplus \overline{L_{[2]}} \oplus \ldots \overline{\oplus L_{[n]}} \text{ and } \overline{L_{[n]}} = L_{(n)},$$

resulting

$$\mathcal{H} = \bigoplus_{n\in\mathbb{N}} L_{(n)} = \bigoplus_{n\in\mathbb{N}} \overline{L_{[n]}}.$$

Corollary

The operator V defined initially on each $L_{(k)}$, $k \in \mathbb{N}$, by

$$V|_{L_{(k)}} = \frac{1}{\sqrt{k}}\, \mathfrak{a}^\dagger|_{L_{(k)}} : L_{(k)} \longrightarrow L_{(k+1)}$$

extends by continuity to a pure isometry on $\mathcal{H}$, with $(\mathrm{Im}\, V)^\perp = \ker V^* = L_{(1)}$.
Its adjoint operator V^* is given by the following action on the subspaces $L_{(k)}$

$$V^*|_{L_{(k)}} = \begin{cases} \frac{1}{\sqrt{k-1}}\, \mathfrak{a}|_{L_{(k)}} : L_{(k)} \longrightarrow L_{(k-1)}, & k > 1 \\ \mathfrak{a}|_{L_{(1)}} : L_{(1)} \longrightarrow \{0\}, & k = 1 \end{cases}.$$

Corollary

The operators $\mathfrak{a}$ *and* $\mathfrak{a}^\dagger$ *admit the extension to the common domain*

$$\mathcal{D}^{\#}_{ext} = \left\{ h = \sum_{k\in\mathbb{N}} h_k \in \mathcal{H} : \ h_k \in L_{(k)} \quad \text{and} \quad \sum_{k\in\mathbb{N}} k\|h_k\|^2 < \infty \right\}$$

by

$$\mathfrak{a} : \sum_{k\in\mathbb{N}} h_k \longmapsto \sum_{k\in\mathbb{N}} \sqrt{k} V^* h_k \quad \text{and} \quad \mathfrak{a}^\dagger : \sum_{k\in\mathbb{N}} h_k \longmapsto \sum_{k\in\mathbb{N}} \sqrt{k} V h_k,$$

on which they are mutually adjoint.

Lemma

In case $\mathfrak{b} = \mathfrak{a}^\dagger$, *we have*

$$\begin{aligned} \ker \mathfrak{a}^n &= \{h \in \mathcal{H} : \mathfrak{a}^n h = 0\} \\ &= L_n = \left\{ h \in \mathcal{H} : h = \sum_{k=0}^{n-1} (\mathfrak{a}^\dagger)^k g_k, \quad g_k \in \ker \mathfrak{a} \right\}. \end{aligned}$$

Theorem

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. Then the following statements are equivalent:

1. *there is a pure isometry V in $\mathcal{H}$;*
2. *the Hilbert space $\mathcal{H}$ admits the orthogonal sum decomposition*

$$\mathcal{H} = \bigoplus_{k=1}^{\infty} \mathcal{H}_{(k)},$$

   where all $\mathcal{H}_{(k)}$ have the same (finite or infinite) dimension;
3. *there are two formally adjoint lowering and raising operators $\mathfrak{a}$ and $\mathfrak{a}^\dagger$, that act invariantly on a common domain $\mathcal{D}$ dense in $\mathcal{H}$, such that the following commutation relation holds*

$$[\mathfrak{a}, \mathfrak{a}^\dagger] = I,$$

   the set $L_{(1)} = \ker \mathfrak{a}$ is a closed subspace of $\mathcal{H}$, and the linear span of the spaces $L_{(n)} := (\mathfrak{a}^\dagger)^{n-1} L_{(1)}$ is dense in $\mathcal{H}$.

Theorem continuation

Moreover, the subspaces $\mathcal{H}_{(k)}$ *in the decomposition*

$$\mathcal{H} = \bigoplus_{k=1}^{\infty} \mathcal{H}_{(k)},$$

are related to the operators V, $\mathfrak{a}$ *and* $\mathfrak{a}^\dagger$ *as follows*

$$\begin{aligned} \mathcal{H}_{(1)} &= \ker V^* = \ker \mathfrak{a} \\ \mathcal{H}_{(k)} &= V^{k-1}(\ker V^*) = (\mathfrak{a}^\dagger)^{k-1}(\ker \mathfrak{a}), \quad \text{for} \quad k > 1. \end{aligned}$$

As a consequence, each one of the theorem items can be used (and will be used) in the characterization of the polyanalytic type spaces.

In particular, the Fock space construction can be done starting with a separable Hilbert space $\mathcal{H}$ and a pure isometry V acting on it.

Introduce the closed subspaces $L_{(k)} := V^{k-1}(\ker V^*)$ and define the operators $\mathfrak{a}$ and $\mathfrak{a}^\dagger$ by their action on subspaces $L_{(k)}$ as the weighted versions of the operators V^* and V:

$$\mathfrak{a} : h_k \in L_{(k)} \longmapsto \begin{cases} \sqrt{k-1}V^* h_k, & k > 1 \\ 0, & k = 1 \end{cases},$$

$$\mathfrak{a}^\dagger : h_k \in L_{(k)} \longmapsto \sqrt{k} V h_k, \quad k \in \mathbb{N},$$

and extend then $\mathfrak{a}$ and $\mathfrak{a}^\dagger$ to the domain $\mathcal{D}_{ext}$, which consists of finite linear combinations of elements from all $L_{(k)}$.

It is well known that a pure isometry is determined up to a unitary equivalence by its multiplicity, i.e., two pure isometries V_1 and V_2, acting on separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, are unitary equivalent, that is there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $V_2 = UV_1U^*$, if and only if

$$\dim \ker V_1^* = \dim \ker V_2^*.$$

That is, all, up to a unitary equivalence, extended Fock space constructions are determined by a singe parameter

$$\mathsf{d} = \dim \ker \mathfrak{a} = \dim \ker V^* \in \overline{\mathbb{N}} = \mathbb{N} \cup \infty.$$

We denote by $\mathfrak{F}_{\mathsf{d}}$ any of the extended Fock spaces constructed in the described way and corresponding to the parameter d.

We will propose later on a canonical, in a sense, representation of $\mathfrak{F}_{\mathsf{d}}$ for each $\mathsf{d} \in \overline{\mathbb{N}}$.

The case of $\mathsf{d} = 1$ returns us to the classical Fock space formalism.

Operator $\mathfrak{a}\mathfrak{a}^\dagger$

Given $\mathsf{d} \in \overline{\mathbb{N}}$, let us consider the extended Fock space $\mathfrak{F}_\mathsf{d}$ in its realization on a separable Hilbert space $\mathcal{H}$ together with the operators $\mathfrak{a}$ and $\mathfrak{a}^\dagger$ with $\dim\ker\mathfrak{a} = \mathsf{d}$.
The operator $\mathfrak{a}\mathfrak{a}^\dagger$ is Hermitian, being defined on $\mathcal{D}_{ext}$, and act therein by the rule

$$\mathfrak{a}\mathfrak{a}^\dagger : \sum_{k\in\mathbb{N}} h_k \longmapsto \sum_{k\in\mathbb{N}} k h_k, \quad \text{where} \quad h_k \in L_{(k)}.$$

The operator $\mathfrak{a}\mathfrak{a}^\dagger$ becomes self-adjoint being extended to its natural domain

$$\mathcal{D}^{\#} = \left\{ h = \sum_{k\in\mathbb{N}} h_k \in \mathcal{H} : \ h_k \in L_{(k)} \quad \text{and} \quad \sum_{k\in\mathbb{N}} \|k h_k\|^2 < \infty \right\}$$

with the same action.

Operator $\mathfrak{a}\mathfrak{a}^\dagger$

We denote by $\mathbf{P}_k = \mathbf{P}_k(\mathrm{d})$ the orthogonal projection of

$$\mathcal{H} = \bigoplus_{n\in\mathbb{N}} L_{(n)}$$

onto $L_{(k)}$. Note that, for each k, $\dim \mathrm{Range}\, \mathbf{P}_k = \mathrm{d}$.
Then the operator $\mathfrak{a}\mathfrak{a}^\dagger$ admits the representation

$$\mathfrak{a}\mathfrak{a}^\dagger = \sum_{k\in\mathbb{N}} k\mathbf{P}_k,$$

understood in the strong operator topology.
This implies immediately the the spectrum of $\mathfrak{a}\mathfrak{a}^\dagger$ consists only on eigenvalues of multiplicity d, $\mathrm{sp}\,(\mathfrak{a}\mathfrak{a}^\dagger) = \mathbb{N}$, and the eigenspace, that corresponds to k, is $L_{(k)}$.

Example: Harmonic Oscillator

It is the operator

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right),$$

densely defined in $L_2(\mathbb{R})$.
Consider the following representation of $\mathfrak{F}_1$: $\mathcal{H} = L_2(\mathbb{R})$, $\mathfrak{a} = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)$ and $\mathfrak{a}^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)$. Then,

$$H = \mathfrak{a}\mathfrak{a}^\dagger - \tfrac{1}{2}I = \sum_{k\in\mathbb{N}}(k - \tfrac{1}{2})\mathbf{P}_k,$$

which implies that the spectrum of the operator H consists of infinitely many equidistant eigenvalues,

$$\lambda_k = k - \tfrac{1}{2}\,, \qquad k \in \mathbb{N}\,,$$

and the corresponding eigenspaces are $H_{k-1}(x)e^{-\frac{x^2}{2}}$, where H_n are Hermite polynomials.

Example: Landau magnetic Hamiltonian

It is the operator

$$\widetilde{\Delta} \;=\; -\frac{\partial^2}{\partial z\,\partial\overline{z}} + \overline{z}\,\frac{\partial}{\partial\overline{z}}\;,$$

densely defined in $L_2(\mathbb{C}, d\mu)$ with the Gaussian measure $d\mu(z) = \frac{1}{\pi} e^{-|z|^2} dxdy$.
Consider the following representation of $\mathfrak{F}_\infty$: $\mathcal{H} = L_2(\mathbb{C}, d\mu)$, $\mathfrak{a} = \frac{\partial}{\partial\overline{z}}$ and $\mathfrak{a}^\dagger = -\frac{\partial}{\partial z} + \overline{z}$.
Then,

$$\widetilde{\Delta} = \mathfrak{a}\mathfrak{a}^\dagger - I = \sum_{k\in\mathbb{N}} (k-1)\mathbf{P}_k,$$

which implies that the spectrum of the operator $\widetilde{\Delta}$ consists of infinitely many equidistant eigenvalues, each of infinite multiplicity (Landau levels), they are of the form

$$\lambda_k = k - 1\;, \qquad k \in \mathbb{N}\;,$$

and the corresponding eigenspaces are $L_{(k)} = F^2_{(k)}(\mathbb{C})$, the true-$k$-poly-Fock spaces.

Example: Perturbed Landau magnetic Hamiltonian

Given $\omega < -\frac{1}{2}$, it is the operator

$$\widetilde{\Delta}_\omega = \frac{1}{1-2\omega}\left\{-\frac{\partial^2}{\partial z\,\partial\overline{z}} + (1-\omega)\overline{z}\,\frac{\partial}{\partial\overline{z}} + \omega z\frac{\partial}{\partial z} + \omega(1-\omega)z\overline{z} + \omega\right\},$$

densely defined in $L_2(\mathbb{C}, d\mu)$ with the Gaussian measure $d\mu(z) = \frac{1}{\pi}e^{-|z|^2}dxdy$.

Consider the following representation of $\mathfrak{F}_\infty$: $\mathcal{H} = L_2(\mathbb{C}, d\mu)$, $\mathfrak{a}_\omega = \frac{1}{\sqrt{1-2\omega}}\left(\frac{\partial}{\partial\overline{z}} - \omega z\right)$ and $\mathfrak{a}_\omega^\dagger = \frac{1}{\sqrt{1-2\omega}}\left(-\frac{\partial}{\partial z} + (1-\omega)\overline{z}\right)$.

Then,

$$\widetilde{\Delta}_\omega = \mathfrak{a}_\omega\mathfrak{a}_\omega^\dagger - I = \sum_{k\in\mathbb{N}}(k-1)\mathbf{P}_k,$$

which implies that the spectrum of the operator $\widetilde{\Delta}_\omega$ consists of infinitely many equidistant eigenvalues, each of infinite multiplicity, they are of the form

$$\lambda_k = k-1\,, \qquad k\in\mathbb{N}\,,$$

and the corresponding eigenspaces are $L_{(k)} = (\mathfrak{a}_\omega^\dagger)^{k-1}(\ker\mathfrak{a}_\omega)$, the true-$k$-poly-Vekua spaces.

Polyanalytic type spaces in $\mathbb{C}$

Fock space

We start with $\mathcal{H} = L_2(\mathbb{C}, d\mu)$ with the Gaussian measure

$$d\mu(z) = e^{-z\cdot\bar{z}} dA(z),$$

and the following lowering and raising operators

$$\mathfrak{a} = \frac{\partial}{\partial \bar{z}}, \qquad \mathfrak{a}^\dagger = -\frac{\partial}{\partial z} + \bar{z}.$$

Recall that the classical Fock space $F^2(\mathbb{C})$ is the closed subspace of $L_2(\mathbb{C}, d\mu)$, which consists of all analytic in $\mathbb{C}$ functions.

Alternatively, it can be defined as the (closed) subspace of all smooth functions satisfying the Cauchy–Riemann equation

$$\mathfrak{a}\,\varphi = \frac{\partial \varphi}{\partial \bar{z}} = 0.$$

Poly-Fock spaces

Introduce also the poly-Fock spaces: for each $k \in \mathbb{N}$, the k-poly-Fock space $F^2_k(\mathbb{C})$ is the closed set of all smooth functions from $L_2(\mathbb{C}, d\mu)$ satisfying the equation

$$\mathfrak{a}^k \varphi = \left(\frac{\partial}{\partial \overline{z}}\right)^k \varphi \;=\; 0.$$

It is convenient to introduce the spaces

$$\begin{aligned} F^2_{(k)}(\mathbb{C}) &= F^2_k(\mathbb{C}) \ominus F^2_{k-1}(\mathbb{C}), \quad \text{for} \;\; k > 1, \\ F^2_{(1)}(\mathbb{C}) &= F^2_1(\mathbb{C}) = F^2(\mathbb{C}), \quad \text{for} \;\; k = 1. \end{aligned}$$

We call the space $F^2_{(k)}(\mathbb{C})$ the true-k-Fock space. It is evident that

$$F^2_k(\mathbb{C}) \;= \bigoplus_{p=1}^{k} F^2_{(p)}(\mathbb{C}).$$

We have then

Proposition

The space $L_2(\mathbb{C}, d\mu)$ *admits the following decomposition*

$$L_2(\mathbb{C}, d\mu) = \bigoplus_{k=1}^{\infty} F^2_{(k)}(\mathbb{C}).$$

An equivalent to the above proposition statement can be formulated in terms of the k-Fock spaces as follows.

Corollary

The set of k*-Fock subspaces* $F^2_k(\mathbb{C})$, $k \in \mathbb{N}$, *of the space* $L_2(\mathbb{C}, d\mu)$ *forms an infinite flag in* $L_2(\mathbb{C}, d\mu)$

$$F^2_1(\mathbb{C}) \subset F^2_2(\mathbb{C}) \subset \ldots \subset F^2_k(\mathbb{C}) \subset \ldots \subset \bigcup_{k=1}^{\infty} F^2_k(\mathbb{C}) \subset L_2(\mathbb{C}, d\mu),$$

and

$$L_2(\mathbb{C}, d\mu) = \overline{\bigcup_{k=1}^{\infty} F^2_k(\mathbb{C})}.$$

Properties of the poly-Fock spaces

Proposition

For each $k = 2, 3, \ldots$, *the operator*

$$\frac{1}{\sqrt{k-1}}\, \mathfrak{a}^{\dagger}|_{F^2_{(k-1)}(\mathbb{C})} : F^2_{(k-1)}(\mathbb{C}) \longrightarrow F^2_{(k)}(\mathbb{C})$$

is an isometric isomorphism, and the operator

$$\frac{1}{\sqrt{k-1}}\, \mathfrak{a}|_{F^2_{(k)}(\mathbb{C})} : F^2_{(k)}(\mathbb{C}) \longrightarrow F^2_{(k-1)}(\mathbb{C}) \ ,$$

is its inverse.

And for each $k \in \mathbb{N}$, the operator

$$\mathbf{A}_{(k)} := \frac{1}{\sqrt{(k-1)!}} \left(\mathfrak{a}^\dagger\right)^{k-1}|_{F^2(\mathbb{C})} : F^2(\mathbb{C}) \longrightarrow F^2_{(k)}(\mathbb{C}) \,,$$

gives an isometric isomorphism between the Fock space $F^2(\mathbb{C})$ and the true-poly-Fock space $F^2_{(k)}(\mathbb{C})$.

Corollary

Each function $\psi(z, \overline{z})$ from the true-k-Fock space $F^2_{(k)}(\mathbb{C})$ is uniquely defined by a function $\varphi(z) \in F^2(\mathbb{C})$ and has the form

$$\psi(z) = \psi(z, \overline{z}) = \sum_{m=0}^{k-1} (-1)^m \frac{\sqrt{(k-1)!}}{m!\,(k-1-m)!}\, \overline{z}^{\,k-1-m}\, \varphi^{(m)}(z),$$

where $\varphi^{(m)}$ is the m-th derivative of the (analytic) function φ, and

$$\|\psi\|_{F^2_{(k)}(\mathbb{C})} = \|\varphi\|_{F^2(\mathbb{C})}.$$

Properties of the poly-Fock spaces

Theorem

Each function $\varphi(z,\overline{z}) \in F^2_k(\mathbb{C})$ *is uniquely defined by* k *functions* $f_1(z)$, ..., $f_k(z)$ *from the Fock space* $F^2(\mathbb{C})$ *and admits the representation*

$$\varphi(z,\overline{z}) = \sum_{\ell=1}^{k} \overline{z}^{\ell-1} \cdot \varphi_\ell(z),$$

where the analytic in $\mathbb{C}$ *functions* $\varphi_\ell(z)$ *have the form*

$$\varphi_\ell(z) = \sum_{p=\ell}^{k} (-1)^{p-\ell} \frac{\sqrt{(p-1)!}}{(p-\ell)!\,(\ell-1)!}\, f_p^{(p-\ell)}(z).$$

Note that $\varphi_k(z) = \frac{f_k(z)}{\sqrt{(k-1)!}} \in F^2(\mathbb{C})$, while the others $\varphi_\ell(z)$, with $\ell = 1, 2, \ldots, k-1$, generically do not belong to $F^2(\mathbb{C})$.

In terms of basis elements

As known, the elements $e_{p,q}$, with $p, q \in \mathbb{Z}_+$, of the orthonormal basis in $L_2(\mathbb{C}, d\mu)$ are polynomials in z and $\overline{z}$, whole leading term is a multiple of $z^p \overline{z}^q$, and are given by

$$e_{p,q}(z, \overline{z}) = \sqrt{p!q!} \sum_{k=0}^{\min\{p,q\}} \frac{(-1)^k}{k!(p-k)!(q-k)!} z^{p-k} \overline{z}^{q-k}.$$

Apart of the operators $\mathfrak{a}$ and $\mathfrak{a}^\dagger$, we introduce also formally adjoint operators

$$\widetilde{\mathfrak{a}} = \frac{\partial}{\partial z}, \qquad \widetilde{\mathfrak{a}}^\dagger = -\frac{\partial}{\partial \overline{z}} + z, \quad \text{with} \quad [\widetilde{\mathfrak{a}}, \widetilde{\mathfrak{a}}^\dagger] = I,$$

that act invariantly on the common dense in $L_2(\mathbb{C}, d\mu)$ domain $\mathcal{D}_0$, being the span of all polynomials $z^p \overline{z}^q$ or, equivalently, the span of all basis elements $e_{p,q}$, with $p, q \in \mathbb{Z}_+$.

Note that the first basis element $e_{0,0} = \mathbf{1}$, is the function identically equals to 1, and all other basis elements are connected with it by

$$e_{p,q} = \frac{1}{\sqrt{p!q!}}(\mathfrak{a}^\dagger)^q(\widetilde{\mathfrak{a}}^\dagger)^p e_{0,0} = \frac{1}{\sqrt{p!q!}}(\widetilde{\mathfrak{a}}^\dagger)^p(\mathfrak{a}^\dagger)^q e_{0,0}.$$

Introduce then the anti-Fock space $\widetilde{F}^2$, which consists of all anti-analytic functions from $L_2(\mathbb{C}, d\mu)$, being $\ker \widetilde{\mathfrak{a}}$.
For all integers $n > 1$ we introduce also the n-poly-anti-Fock spaces $\widetilde{F}^2_n$, which consist of all n-anti-polyanalytic functions from $L_2(\mathbb{C}, d\mu)$ and are $\ker \widetilde{\mathfrak{a}}^n$.
It is easy to figure out, these spaces, in therms of basis elements $e_{p,q}$, admit the following representations

$$\begin{aligned}
F^2 = F^2_1 &= \overline{\operatorname{span}}\{e_{p,q} : p \in \mathbb{Z}_+, \quad q = 0\},\\
F^2_n &= \overline{\operatorname{span}}\{e_{p,q} : p \in \mathbb{Z}_+, \quad q = 0, 1, \ldots, n-1\},\\
\widetilde{F}^2 = \widetilde{F}^2_1 &= \overline{\operatorname{span}}\{e_{p,q} : p = 0, \quad q \in \mathbb{Z}_+\},\\
\widetilde{F}^2_n &= \overline{\operatorname{span}}\{e_{p,q} : p = 0, 1, \ldots, n-1, \quad q \in \mathbb{Z}_+\}.
\end{aligned}$$

Representation of generalised Fock spaces $\mathfrak{F}_{\mathsf{d}}$

Given $\mathsf{d} \in \overline{\mathbb{N}}$, our canonical realization of the generalised Fock spaces $\mathfrak{F}_{\mathsf{d}}$ is given as follows.
We set $\mathcal{H}_{\mathsf{d}} = \widetilde{F}^2_{\mathsf{d}}$ (the set of polynomials in z of degree $\mathsf{d}-1$ with anti-analytic coefficients), if d is finite, and $\mathcal{H}_\infty = L_2(\mathbb{C}, d\mu)$, if d is infinite, and introduce

$$\mathfrak{a} = \frac{\partial}{\partial \overline{z}}, \qquad \mathfrak{a}^\dagger = -\frac{\partial}{\partial z} + \overline{z},$$

which are mutually adjoint being defined on

$$\mathcal{D}^{\#}_{ext}(\mathsf{d}) = \left\{ f = \sum_{k\in\mathbb{N}} f_k : \, f_k \in F^2_{(k)} \cap \mathcal{H}_{\mathsf{d}} \;\text{ and }\; \sum_{k\in\mathbb{N}} k\|f_k\|^2 < \infty \right\}.$$

Note that if $\mathsf{d} = 1$, this is exactly the already considered Berezin example: $\mathcal{H} = \widetilde{F}^2(\mathbb{C})$ is the space of all anti-analytic functions in $L_2(\mathbb{C}, d\mu)$, and $\mathfrak{a} = \frac{\partial}{\partial \overline{z}}$, $\mathfrak{a}^\dagger = \overline{z}$ with $\Phi_0 = 1$.

Generalised polyanalytic function spaces

Given $\omega \in \mathbb{R}$, we start with two, invariantly acting on the linear set of all smooth in $\mathbb{C}$ functions, operators

$$\mathfrak{a} = \frac{\partial}{\partial \overline{z}} - \omega z, \quad \text{and} \quad \mathfrak{b} = \overline{z}, \quad \text{with} \quad [\mathfrak{a}, \mathfrak{b}] = I.$$

The set of all functions f, satisfying $\mathfrak{a} f = 0$, forms a class of the generalized analytic functions $\mathfrak{A}_\omega := \mathfrak{A}(-\omega z, 0; \mathbb{C})$ (in the Vekua notation), and, as it easily seen, consists of all functions of the form

$$f(z) = e^{\omega z \overline{z}} \varphi(z), \quad \text{with analytic} \quad \varphi(z).$$

Then n-poly-generalized analytic functions (those satisfying $\mathfrak{a}^n g = 0$) are of the form

$$g = \sum_{k=0}^{n-1} \overline{z}^k f_k, \quad \text{with} \quad f_k \in \mathfrak{A}_\omega,$$

or

$$g = e^{\omega z \overline{z}} \varphi_n(z), \quad \text{with} \quad n\text{-poly-analytic} \quad \varphi_n(z).$$

Poly-Vekua spaces

We define then the Vekua space (ω-Vekua space) by

$$\mathcal{V}^2 := \mathcal{V}^2(\omega, \mathbb{C}) = L_2(\mathbb{C}, d\mu) \cap \mathfrak{A}(-\omega z, 0; \mathbb{C}).$$

Calculate

$$\|f\|^2_{L_2(\mathbb{C}, d\mu)} = \|e^{\omega z \bar{z}} \varphi(z)\|^2 = \int_{\mathbb{C}} |\varphi(z)|^2 e^{-(1-2\omega)|z|^2} dA(z).$$

That is $\mathcal{V}^2$ is non-trivial if and only if $\omega < \frac{1}{2}$, and $f \in \mathcal{V}^2$ if and only if $\varphi \in F^2(\mathbb{C}, d\mu_{1-2\omega})$.

Introduce also the n-poly-Vekua space $\mathcal{V}^2_n$ as the set of all functions g in $L_2(\mathbb{C}, d\mu)$ that satisfy the equation $\mathfrak{a}^n g = 0$.

Then $g = e^{\omega z \bar{z}} \varphi_n(z) \in \mathcal{V}^2_n$ if and only if $\varphi_n(z) \in F^2_n(\mathbb{C}, d\mu_{1-2\omega})$.

Isomorphism

Introduce the unitary operator $U_\omega : L_2(\mathbb{C}, d\mu) \to L_2(\mathbb{C}, d\mu_{1-2\omega})$ by

$$U_\omega : f \longmapsto \frac{1}{\sqrt{1-2\omega}} e^{-\omega|z|^2} f,$$

its inverse $U_\omega^{-1} : L_2(\mathbb{C}, d\mu_{1-2\omega}) \to L_2(\mathbb{C}, d\mu)$ is given by

$$U_\omega^{-1} : h \longmapsto \sqrt{1-2\omega}\, e^{\omega|z|^2} h.$$

Lemma

For each $n \in \mathbb{N}$, the operator U_ω gives the isomertical isomorphism between the n-polyanalytic type spaces: $\mathcal{V}_n^2$ and $F_n^2(\mathbb{C}, d\mu_{1-2\omega})$:

$$U_\omega : \mathcal{V}_n^2 \longrightarrow F_n^2(\mathbb{C}, d\mu_{1-2\omega}).$$

Recall that the operators

$$\mathbf{a}_{1-2\omega} = \frac{1}{\sqrt{1-2\omega}} \frac{\partial}{\partial \overline{z}} \quad \text{and} \quad \mathbf{a}_{1-2\omega}^\dagger = -\frac{1}{1-2\omega} \frac{\partial}{\partial z} + \sqrt{1-2\omega}\, \overline{z}$$

are formally adjoint in $L_2(\mathbb{C}, d\mu_{1-2\omega})$, being defined on the span of all poly-Fock spaces $F_n^2(\mathbb{C}, d\mu_{1-2\omega})$.

Corollary

The operators

$$\begin{aligned} \mathfrak{a}_\omega &= U_\omega^{-1}\mathbf{a}_{1-2\omega}U_\omega = \frac{1}{\sqrt{1-2\omega}}\left(\frac{\partial}{\partial\overline{z}} - \omega\, z\right) \\ \mathfrak{a}_\omega^\dagger &= U_\omega^{-1}\mathbf{a}_{1-2\omega}^\dagger U_\omega = \frac{1}{\sqrt{1-2\omega}}\left(-\frac{\partial}{\partial z} + (1-\omega)\overline{z}\right) \end{aligned}$$

are formally adjoint in $L_2(\mathbb{C}, d\mu)$, *being defined on the span of all poly-Vekua spaces* $\mathcal{V}_n^2$.

Each function $\psi(z,\overline{z})$ *from the true-*n*-poly-Vekua space* $\mathcal{V}_{(n)}^2 = \mathcal{V}_n^2 \ominus \mathcal{V}_{n-1}^2$ *is uniquely defined by a function* $\varphi(z) \in \mathcal{V}^2$ *and has the form*

$$\psi(z) = \psi(z,\overline{z}) = (\mathfrak{a}_\omega^\dagger)^{n-1}\varphi(z)$$

Furthermore

$$L_2(\mathbb{C}, d\mu) = \bigoplus_{n\in\mathbb{N}} \mathcal{V}_{(n)}^2.$$

Polyanalytic type spaces in $\mathbb{D}$

Poly-Bergman spaces

For each $\lambda > -1$, introduce the probability measure

$$d\nu_\lambda = (\lambda+1)\,(1-z\overline{z})^\lambda\, dA(z), \quad \text{where} \quad dA(z) = \frac{1}{\pi} dxdy,$$

and then the Hilbert space $L_2(\mathbb{D}, d\nu_\lambda)$. Recall that the system of functions

$$e_{p,q}^{(\lambda)} := e_{p,q}^{(\lambda)} e_{p,q}^{(\lambda)} = \sqrt{\frac{(\lambda+p+q+1)p!q!}{(\lambda+1)\Gamma(\lambda+p+1)\Gamma(\lambda+q+1)}}$$

$$\times \sum_{k=0}^{\min\{p,q\}} (-1)^k \frac{\Gamma(\lambda+p+q+1-k)}{k!(p-k)!(q-k)!} z^{p-k}\overline{z}^{q-k},$$

with $p, q \in \mathbb{Z}_+$, forms an orthonormal basis in $L_2(\mathbb{D}, d\nu_\lambda)$.
We mention also that $\|z^k\overline{z}^\ell\|^2 = \|z^{k+\ell}\|\|^2$ in $L_2(\mathbb{D}, d\mu_\lambda)$ is given by

$$\|z^k\overline{z}^\ell\|^2 = \frac{\Gamma(\lambda+2)(k+\ell)!}{\Gamma(\lambda+k+\ell+2)}.$$

Poly-Bergman spaces

On the dense in $L_2(\mathbb{D}, d\nu_\lambda)$ domain

$$\mathcal{D}_{\mathfrak{a}} = \left\{ f \in L_2(\mathbb{D}, d\nu_\lambda) : \ \frac{\partial f}{\partial \overline{z}} \in L_2(\mathbb{D}, d\mu_\lambda) \right\}$$

introduce the operators

$$\mathfrak{a} = \frac{\partial}{\partial \overline{z}} \qquad \text{and} \qquad \mathfrak{b} = \overline{z},$$

which obviously satisfy the relation $[\mathfrak{a}, \mathfrak{b}] = I$.
Note that the operator $\overline{z}$ is bounded in $L_2(\mathbb{D}, d\nu_\lambda)$.
Introduce now the spaces

$$L_{[1]} = \ker \mathfrak{a} = \left\{ f \in L_2(\mathbb{D}, d\nu_\lambda) : \ \mathfrak{a} f = \frac{\partial f}{\partial \overline{z}} = 0 \right\},$$

which coincides with the standard weighted Bergman space $\mathcal{A}^2_\lambda$,
and

$$L_{[n]} = \mathfrak{b}^{n-1} L_{[1]} = \left\{ \overline{z}^{n-1} f : \ f \in \mathcal{A}^2_\lambda \right\} = \overline{z}^{n-1} \mathcal{A}^2_\lambda = \mathcal{A}^2_{\lambda,[n]}, \qquad n \in \mathbb{N}.$$

Observe that the set $\mathcal{D}_0$, formed by finite linear combinations of elements from the spaces $L_{[n]} := \mathfrak{b}^{n-1} L_{[1]}$, $n \in \mathbb{N}$, is dense in $L_2(\mathbb{D}, d\nu_\lambda)$, and that the operators $\mathfrak{a}$ and $\mathfrak{b}$ act invariantly in $\mathcal{D}_0$.

As known, the Bergman space $\mathcal{A}^2_\lambda = L_{[1]}$ is closed, and it is easy to figure out that all subsequent subspaces $L_{[n]} = \overline{z}^{n-1} \mathcal{A}^2_\lambda$ are also closed in $L_2(\mathbb{D}, d\nu_\lambda)$.

Surprise: the sum $L_{[1]} \dot{+} L_{[2]} \dot{+} \ldots \dot{+} L_{[n]}$ of closed subspaces is not closed. In fact we have even more strong result.

Lemma

For each $n > 1$, the direct sum of the two closed subspaces

$$L_{n-1} \dot{+} L_{[n]} = \operatorname{clos} \left(L_{[1]} \dot{+} L_{[2]} \dot{+} \ldots \dot{+} L_{[n-1]} \right) \dot{+} L_{[n]}$$

is not closed.

Illustrative example

For the closed subspaces $\mathcal{A}^2$, $\overline{z}\mathcal{A}^2$, and 2-poly-Bergman space $\mathcal{A}^2_2$ of the unweighted space $L_2(\mathbb{D}, dA)$, we have

$$\mathcal{A}^2 + \overline{z}\mathcal{A}^2 \subsetneq \operatorname{clos}(\mathcal{A}^2 + \overline{z}\mathcal{A}^2) = \mathcal{A}^2_2.$$

To show this we represent first $\mathcal{A}^2_2 = \mathcal{A}^2 \oplus \mathcal{A}^2_{(2)}$ as the orthogonal sum of two true-poly-Bergman spaces, where

$$\mathcal{A}^2_{(2)} = \overline{\operatorname{span}}\,\{e_{p,1} : p \in \mathbb{Z}_+\},$$

with $e_{p,1}$ for $\lambda = 0$. Observe that

$$e_{p,1} = \sqrt{p+2}\left[(p+1)\overline{z}z^p - pz^{p-1}\right] = (p+1)e_p^{(1)} - \sqrt{p(p+2)}e_{p-1,0},$$

where $e_{p-1,0}$ and $e_p^{(1)}$ are the basis elements of $\mathcal{A}^2$ and $\overline{z}\mathcal{A}^2$, respectively.

It is sufficient now to give an example of a function from $\mathcal{A}^2_{(2)} \subset \operatorname{clos}(\mathcal{A}^2 \dotplus \overline{z}\mathcal{A}^2)$ which does not belong to $\mathcal{A}^2 \dotplus \overline{z}\mathcal{A}^2$.

Indeed, consider

$$f = \sum_{p\in\mathbb{N}} \frac{1}{p+1} e_{p,1} = \sum_{p\in\mathbb{N}} e_p^{(1)} - \sum_{p\in\mathbb{N}} \frac{\sqrt{p(p+2)}}{p+1} e_{p-1,0}.$$

Then the sequence of functions

$$f_n = \sum_{p=1}^{n} \frac{1}{p+1} e_{p,1} = \sum_{p=1}^{n} e_p^{(1)} - \sum_{p=1}^{n} \frac{\sqrt{p(p+2)}}{p+1} e_{p-1,0} = f_n^{(1)} - f_n^{(2)}, \quad n \in \mathbb{N},$$

converges in norm to $f \in \mathcal{A}^2_{(2)} \subset \mathcal{A}^2_2$, but none of $f_n^{(1)}$ and $f_n^{(2)}$, $n \in \mathbb{N}$, converges in $\mathcal{A}^2$ and $\overline{z}\mathcal{A}^2$, respectively.

Relations among poly-Bergman spaces

We have

$$\mathcal{A}^2_{\lambda,n-1}\dot{+}\mathcal{A}^2_{\lambda,[n]} = \mathcal{A}^2_{\lambda,n-1}\dot{+}\overline{z}^{n-1}\mathcal{A}^2_{\lambda} \subsetneq \mathcal{A}^2_{\lambda,n} = \operatorname{clos}\left(\mathcal{A}^2_{\lambda,n-1}\dot{+}\mathcal{A}^2_{\lambda,[n]}\right),$$

while

$$\mathcal{A}^2_{\lambda,n-1} \oplus \mathcal{A}^2_{\lambda,(n)} = \mathcal{A}^2_{\lambda,n}.$$

Thus we have two different forms of the representation of $L_2(\mathbb{D}, d\nu_\lambda)$ via polyanalytic functions:

$$L_2(\mathbb{D}, d\mu_\lambda) = \operatorname{clos}\left(\bigcup_{n\in\mathbb{N}} \mathcal{A}^2_{\lambda,n}\right) \quad \text{and} \quad L_2(\mathbb{D}, d\mu_\lambda) = \bigoplus_{n\in\mathbb{N}} \mathcal{A}^2_{\lambda,(n)}.$$

True-poly-Bergman spaces

Consider the unweighted case of $\mathcal{H} = L_2(\mathbb{D}, dA(z))$, and recall that the system of functions

$$e_{p,q}(z,\overline{z}) = \sqrt{p+q+1} \sum_{k=0}^{\min\{p,q\}} (-1)^k \frac{(p+q-k)!}{k!(p-k)!(q-k)!} z^{p-k}\overline{z}^{q-k},$$

with $p,\ q \in \mathbb{Z}_+$, forms its orthonormal basis.

So far we have the following description of the true-poly-Bergman spaces

$$\mathcal{A}^2_{(n)} = \mathcal{A}^2_n \ominus \mathcal{A}^2_{n-1},$$

where $\mathcal{A}^2_n = \mathrm{clos}(L_{[1]} \dot{+} L_{[2]} \dot{+} \ldots \dot{+} L_{[n]})$, obtained via not formally adjoint operators $\frac{\partial}{\partial \overline{z}}$ and $\overline{z}$, and

$$\mathcal{A}^2_{(n)} = \overline{\mathrm{span}}\,\{e_{p,q} : p \in \mathbb{Z}_+, \quad q = n-1\}$$

On the operators $\mathfrak{a}$ and $\mathfrak{a}^\dagger$

Our aim is to find formally adjoint operators $\mathfrak{a}$ and $\mathfrak{a}^\dagger$, such that $[\mathfrak{a}, \mathfrak{a}^\dagger] = I$ and $\ker \mathfrak{a} = \mathcal{A}^2$, being the Bergman space.
Then,

$$\mathcal{A}^2_{(n)} = (\mathfrak{a}^\dagger)^{n-1}(\ker \mathfrak{a}) = (\mathfrak{a}^\dagger)^{n-1}\mathcal{A}^2.$$

Following A. Wünsche '2005, on the dense in $L_2(\mathbb{D}, dA)$ domain $\mathcal{D}$, which consists of all finite linear combinations of the basis elements $e_{p,q}$, or, which is the same, all finite linear combinations of the monomials $m_{p,q}$, we define the operators

$$\begin{aligned} L &= z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial z}z - \frac{\partial}{\partial \overline{z}}\overline{z}, \\ H &= -4\frac{\partial^2}{\partial z \partial \overline{z}} + \left(z\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}\overline{z}\right)^2. \end{aligned}$$

The operators L and H act on the basis elements as follows

$$Le_{p,q} = (p-q)e_{p,q} \qquad \text{and} \qquad He_{p,q} = (p+q+1)^2 e_{p,q}.$$

All formulas characterizing the action of the involved operators on the basis elements can be verified by the direct calculations.
All eigenvalues of the diagonal operator H are positive, thus all powers H^s, with $s \in \mathbb{R}$, are well defined, in particular

$$H^{\frac{1}{2}} e_{p,q} = (p+q+1)e_{p,q}.$$

Then introduce the diagonal operator, densely defined on $\mathcal{D}$,

$$L_{(2)} = \frac{1}{2}\left(H^{\frac{1}{2}} - L - 1\right) = \frac{1}{2}\left(H^{\frac{1}{2}} + \overline{z}\frac{\partial}{\partial \overline{z}} - \frac{\partial}{\partial z} z\right),$$

which acts of the basis elements by $L_{(2)} e_{p,q} = q e_{p,q}$.
Note that $\ker L_{(2)} = \mathcal{A}^2$, while on $(\mathcal{A}^2)^\perp = L_2(\mathbb{D}, dA) \ominus \mathcal{A}^2$ the operator $L_{(2)}$ is diagonal, whose all eigenvalues are positive.

We will use then $L_{(2)}^{-1}$ and $L_{(2)}^{-\frac{1}{2}}$, being well defined on $(\mathcal{A}^2)^\perp$.

Operators $\mathfrak{a}$ and $\mathfrak{a}^\dagger$

We also need the following operators, densely defined on $\mathcal{D}$,

$$K_-^{(2)} = H^{\frac{1}{4}}\left[zL_{(2)} + (1-z\overline{z})\frac{\partial}{\partial \overline{z}}\right]H^{-\frac{1}{4}}$$
$$K_+^{(2)} = H^{\frac{1}{4}}\left[\overline{z}L_{(2)} - \frac{\partial}{\partial z}(1-z\overline{z})\right]H^{-\frac{1}{4}}.$$

These operators act on the basis elements as follows

$$K_-^{(2)}e_{p,q} = qe_{p,q-1} \quad \text{and} \quad K_+^{(2)}e_{p,q} = (q+1)e_{p,q+1}.$$

Invariant (basis independent) form of $\mathfrak{a}$ and $\mathfrak{a}^\dagger$, satisfying $[\mathfrak{a}, \mathfrak{a}^\dagger] = I$ is as follows

$$\mathfrak{a} = \begin{cases} K_-^{(2)}L_{(2)}^{-\frac{1}{2}} = H^{\frac{1}{4}}\left[zL_{(2)} + (1-z\overline{z})\frac{\partial}{\partial \overline{z}}\right]H^{-\frac{1}{4}}L_{(2)}^{-\frac{1}{2}}, & \text{on } (\mathcal{A}^2)^\perp \\ 0, & \text{on } \mathcal{A}^2 \end{cases},$$

$$\mathfrak{a}^\dagger = L_{(2)}^{-\frac{1}{2}}K_+^{(2)} = L_{(2)}^{-\frac{1}{2}}H^{\frac{1}{4}}\left[L_{(2)}\overline{z} - \frac{\partial}{\partial z}(1-z\overline{z})\right]H^{-\frac{1}{4}}.$$

Some (counter)examples

The situation on classification of extended Fock spaces changes drastically in case when we do not anymore assume that $\mathfrak{b} = \mathfrak{a}^\dagger$. Considering general $\mathfrak{a}$ and $\mathfrak{b}$ we may not expect that, for two different $\mathcal{H}_1$, $\mathfrak{a}_1$, $\mathfrak{b}_1$ and $\mathcal{H}_2$, $\mathfrak{a}_2$, $\mathfrak{b}_2$ with $\dim\ker\mathfrak{a}_1 = \dim\ker\mathfrak{a}_2$, there exists an invertible operator $S : \mathcal{H}_1 \to \mathcal{H}_2$ such that $S\mathfrak{a}_1 S^{-1} = \mathfrak{a}_2$ and $S\mathfrak{b}_1 S^{-1} = \mathfrak{b}_2$.

Example

Let $\mathcal{H}_1 = \mathcal{H}_2 = L_2(\mathbb{D}, d\mu_\lambda)$, $\mathfrak{a}_1 = \frac{\partial}{\partial\overline{z}}$, $\mathfrak{b}_1 = \overline{z}$ and $\mathfrak{a}_2 = \frac{\partial}{\partial\overline{z}}$, $\mathfrak{b}_2 = -\frac{\partial}{\partial z} + \overline{z}$. The existence of S with $S\overline{z}S^{-1} = -\frac{\partial}{\partial z} + \overline{z}$ is impossible, as the operator in the left-hand side is bounded, while the operator in the right-hand side is unbounded.

Examples

Note that in the above example $\dim\ker \mathfrak{a}_1 = \dim\ker \mathfrak{a}_2 = \infty$, while $0 = \dim\ker \mathfrak{b}_1 \neq \dim\ker \mathfrak{b}_2 = \infty$.
The extra condition $\dim\ker \mathfrak{b}_1 = \dim\ker \mathfrak{b}_2$ does not help much.
In the next example $\dim\ker \mathfrak{a}_1 = \dim\ker \mathfrak{a}_2 = \infty$ and $\dim\ker \mathfrak{b}_1 = \dim\ker \mathfrak{b}_2 = 0$.

Example

Let $\mathcal{H}_1 = L_2(\mathbb{D}, d\mu_\lambda)$, $\mathfrak{a}_1 = \frac{\partial}{\partial \overline{z}}$, $\mathfrak{b}_1 = \overline{z}$ and $\mathcal{H}_2 = L_2(\mathbb{C}, d\mu)$, $\mathfrak{a}_2 = \frac{\partial}{\partial \overline{z}}$, $\mathfrak{b}_2 = \overline{z}$. Again, the existence of S with $S\overline{z}S^{-1} = \overline{z}$ is impossible, as the operator in the left-hand side is bounded, while the operator in the right-hand side is unbounded.

In the previous two examples the operator $\mathfrak{b}_1 = \overline{z}$ was bounded, while the operator $\mathfrak{b}_2$ was unbounded.
In the next example all operators involved are unbounded, and furthermore $\dim\ker \mathfrak{a}_1 = \dim\ker \mathfrak{a}_2 = \infty$ and $\dim\ker \mathfrak{b}_1 = \dim\ker \mathfrak{b}_2 = 0$.

Examples

Example

Let $\mathcal{H}_1 = \mathcal{H}_2 = L_2(\mathbb{C}, d\mu)$, $\mathfrak{a}_1 = \mathfrak{a}_2 = \frac{\partial}{\partial \overline{z}}$, $\mathfrak{b}_1 = \overline{z}$, $\mathfrak{b}_2 = \mathfrak{a}^\dagger = -\frac{\partial}{\partial z} + \overline{z}$. The operators $\mathfrak{a}_{1,2}$ and $\mathfrak{b}_{1,2}$ are densely defined and act invariantly on the domain $\mathcal{D}_0$, being the linear span of all monomials $z^p \overline{z}^q$, with $p, q \in \mathbb{Z}_+$.
Let further

$$\begin{aligned} L_{[1]} &= \ker \mathfrak{a}_1|_{\mathcal{D}_0} = \ker \mathfrak{a}_2|_{\mathcal{D}_0} = \operatorname{span}\{z^p : p \in \mathbb{Z}_+\}, \\ L'_{[2]} &= \mathfrak{b}_1(L_{[1]}) = \operatorname{span}\{\overline{z} z^p : p \in \mathbb{Z}_+\}, \\ L''_{[2]} &= \mathfrak{b}_2(L_{[1]}) = \operatorname{span}\{-p z^{p-1} + \overline{z} z^p : p \in \mathbb{Z}_+\}. \end{aligned}$$

Then $\overline{L_{[1]}} + \overline{L''_{[2]}} = \overline{L_{[1]}} \oplus \overline{L''_{[2]}}$ is a closed subspace of $L_2(\mathbb{C}, d\mu)$, while $\overline{L_{[1]}} + \overline{L'_{[2]}} = \overline{L_{[1]}} \dot{+} \overline{L'_{[2]}}$ is not closed (the minimal angle between $\overline{L_{[1]}}$ and $\overline{L'_{[2]}}$ is zero).

Assuming now the existence of the invertible operator S that connect two extended Fock spaces of this example, we have

$$S^{-1}(\overline{L_{[1]}} + \overline{L''_{[2]}}) = \overline{L_{[1]}} + \overline{L'_{[2]}},$$

where the space in the left-hand side is closed, while the space in the right-hand side is not. Contradiction.

What is preserved?

Lemma

Given two different extended Fock spaces $\mathcal{H}_1$ *with* $\mathfrak{a}_1$, $\mathfrak{b}_1$ *and* $\mathcal{H}_2$ *with* $\mathfrak{a}_2$, $\mathfrak{b}_2$, *for which* $\dim\ker\mathfrak{a}_1 = \dim\ker\mathfrak{a}_2$, *there exists a unitary operator* $U : \mathcal{H}_1 \to \mathcal{H}_2$ *such that*

$$U(L'_n) = U(\mathrm{clos}\,(L'_{[1]} \dot{+} L'_{[2]} \dot{+} \ldots \dot{+} L'_{[n]})) = \mathrm{clos}\,(L''_{[1]} \dot{+} L''_{[2]} \dot{+} \ldots \dot{+} L''_{[n]}) = L''_n,$$

for all $n \in \mathbb{N}$. *Here* $L'_{[n]} = \mathfrak{b}_1^{n-1}(\ker\mathfrak{a}_1)$ *and* $L''_{[n]} = \mathfrak{b}_2^{n-1}(\ker\mathfrak{a}_2)$.

In others words: Under the assumptions of the lemma, there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ which maps each element of the infinite flag

$$L'_1 \subset L'_2 \subset \ldots \subset L'_n \ldots \subset \bigcup_{k\in\mathbb{N}} L'_k \subset \mathcal{H}_1$$

onto the corresponding element of the flag

$$L''_1 \subset L''_2 \subset \ldots \subset L''_n \ldots \subset \bigcup_{k\in\mathbb{N}} L''_k \subset \mathcal{H}_2.$$

Basis oriented approach

Nonstandard weighted Hilbert spaces

Let D be either $\mathbb{D}$ or $\mathbb{C}$, and let J be either $[0,1)$ or $\mathbb{R}_+$, so that $D = J \times \mathbb{T}$, where $\mathbb{T}$ is the unit circle in $\mathbb{C}$.
Let $\mathcal{H}$ be any weighted Hilbert space $L_2(D, d\nu)$, with the probability measure $d\nu(z) = c_\omega\, \omega(|z|) dA(z)$, whose radial weight function $\omega : D \to \mathbb{R}_+$ is such that the linear span of the monomials $m_{p,q} := z^p \overline{z}^q$, for all $p,\, q \in \mathbb{Z}_+$, is dense in $\mathcal{H}$.

Considered: Poly-Bergman: $\mathbb{D}$, $\omega \equiv 1$; poly-Fock: $\mathbb{C}$, $\omega = e^{-|z|^2}$.
Standard: $\mathbb{D}$, $\omega = (1-|z|^2)^\lambda$, $\lambda > -1$; $\quad \mathbb{C}$, $\omega = \alpha e^{-\alpha|z|^2}$, $\alpha > 0$.
Nonstandard: $\mathbb{D}$, $\omega = (1-|z|)^\lambda \exp\left(\frac{-c}{(1-|z|)^\alpha}\right)$, $\lambda \geq 0,\ \alpha, c > 0$;
$\mathbb{C}$, $\omega = \frac{e^{-\alpha|z|^2}}{(1+|z|^2)^t}$, $\alpha > 0$, $t \in \mathbb{R}$, $\quad \omega = |z|^s e^{-\alpha|z|^{2m}}$, $m \geq 1$, $\alpha, s > 0$.

$$\begin{aligned} \langle m_{p,q}, m_{k,\ell} \rangle &= \frac{1}{\pi} \int_0^{2\pi} e^{\theta(p-q+\ell-k)} d\theta \int_J r^{p+q+k+\ell} \omega(r) r dr \\ &= \delta_{p-q,k-\ell} \int_J s^{1/2(p+q+k+\ell)} \omega(\sqrt{s}) ds. \end{aligned}$$

Orthonormal polynomials

For $\xi = |p - q|$, we introduce first the space $L_2(J, d\eta_\xi)$, where $d\eta_\xi(s) = s^\xi \omega(\sqrt{s}) ds$.

Then there exists a sequence $\{P_k^{(\xi)}\}_{k \in \mathbb{Z}_+}$ of the orthonormal polynomials with real coefficients,

$$\langle P_k^{(\xi)}, P_\ell^{(\xi)} \rangle = \int_J P_k^{(\xi)}(s) P_\ell^{(\xi)}(s) s^\xi \omega(\sqrt{s}) ds = \delta_{k,\ell},$$

giving a basis of $L_2(J, d\eta_\xi)$.

Observe that all moments

$$\omega_k^{(\xi)} := \int_J s^k d\eta_\xi(s) = \|z^{k+\xi}\|^2, \qquad k \in \mathbb{Z}_+,$$

are finite.

Orthonormal polynomials

Each polynomial $P_k^{(\xi)}(s)$ is of degree k, with a positive leading coefficient, and has the form

$$P_k^{(\xi)}(s) = \frac{1}{\sqrt{\Delta_{k-1}\Delta_k}} \begin{vmatrix} \omega_0^{(\xi)} & \omega_1^{(\xi)} & \omega_2^{(\xi)} & \cdots & \omega_k^{(\xi)} \\ \omega_1^{(\xi)} & \omega_2^{(\xi)} & \omega_3^{(\xi)} & \cdots & \omega_{k+1}^{(\xi)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \omega_{k-1}^{(\xi)} & \omega_k^{(\xi)} & \omega_{k+1}^{(\xi)} & \cdots & \omega_{2k-1}^{(\xi)} \\ 1 & s & s^2 & \cdots & s^k \end{vmatrix},$$

where

$$\Delta_k = \begin{vmatrix} \omega_0^{(\xi)} & \omega_1^{(\xi)} & \omega_2^{(\xi)} & \cdots & \omega_k^{(\xi)} \\ \omega_1^{(\xi)} & \omega_2^{(\xi)} & \omega_3^{(\xi)} & \cdots & \omega_{k+1}^{(\xi)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \omega_{k-1}^{(\xi)} & \omega_k^{(\xi)} & \omega_{k+1}^{(\xi)} & \cdots & \omega_{2k-1}^{(\xi)} \\ \omega_k^{(\xi)} & \omega_{k+1}^{(\xi)} & \omega_{k+2}^{(\xi)} & \cdots & \omega_{2k}^{(\xi)} \end{vmatrix}$$

is the determinant of the so-called Gram matrix.

Basis on $\mathcal{H}$

Proposition

The collection of functions

$$e_{p,q} = e_{p,q}(rt) = t^{p-q} r^{|p-q|} P^{(|p-q|)}_{\min\{p,q\}}(r^2), \quad z = rt \quad \text{and} \quad p, q \in \mathbb{Z}_+,$$

forms an orthonormal basis on $\mathcal{H}$.

In complex coordinates z and $\overline{z}$, we have

$$e_{p,q}(z, \overline{z}) = \sum_{k=0}^{\min\{p,q\}} a_{p,q;k} z^{p-k} \overline{z}^{q-k},$$

where $a_{p,q;k}$ some calculable coefficients.

Note that $e_{q,p} = \overline{e_{p,q}}$ for all $p, q \in \mathbb{Z}_+$.

Example

Let $\mathcal{H} = L_2(\mathbb{D}, d\nu_\lambda)$. In this case, for each $\xi = |p - q| \in \mathbb{Z}_+$, the orthonormal polynomials $\{P_k^{(\xi)}\}_{k \in \mathbb{Z}_+}$ have to satisfy the relations

$$\langle P_k^{(\xi)}, P_\ell^{(\xi)} \rangle = (\lambda + 1) \int_0^1 P_k^{(\xi)}(s) P_\ell^{(\xi)}(s) s^\xi (1 - s)^\lambda ds = \delta_{k,\ell}.$$

They turn to be the shifted Jacobi polynomials. The corresponding elements of the orthonormal basis are called disk polynomials and are given by

$$\begin{aligned} e_{p,q}^{(\lambda)}(z, \overline{z}) &= \sqrt{\frac{(\lambda + p + q + 1)\, p! q!}{(\lambda + 1)\Gamma(\lambda + p + 1)\Gamma(\lambda + q + 1)}} \\ &\times \sum_{k=0}^{\min\{p,q\}} (-1)^k \frac{\Gamma(\lambda + p + q + 1 - k)}{k!(p - k)!(q - k)!} z^{p-k} \overline{z}^{q-k}. \end{aligned}$$

Example

Let $\mathcal{H} = L_2(\mathbb{C}, d\mu_1)$, where $d\mu_1 = e^{-|z|^2} dA(z)$.
In this case, for each $\xi = |p - q| \in \mathbb{Z}_+$, the orthonormal polynomials $\{P_k^{(\xi)}\}_{k \in \mathbb{Z}_+}$ have to satisfy the relations

$$\langle P_k^{(\xi)}, P_\ell^{(\xi)} \rangle = \int_0^\infty P_k^{(\xi)}(s) P_\ell^{(\xi)}(s) s^\xi e^{-s} ds = \delta_{k,\ell}.$$

They coincide with the classical Laguerre polynomials $L_n^{(\xi)}$.
While the corresponding elements of the orthonormal basis are normalized complex Hermite polynomials and are given by

$$e_{p,q}(z, \overline{z}) = \sqrt{p!q!} \sum_{k=0}^{\min\{p,q\}} \frac{(-1)^k}{k!(p-k)!(q-k)!} z^{p-k} \overline{z}^{q-k}.$$

Example

Let $\mathcal{H} = L_2(\mathbb{C}, d\nu)$, where $d\nu = \frac{2}{\sqrt{\pi}} e^{-|z|^4} dA(z)$.
Formulas for a few first basis elements in $L_2(\mathbb{C}, d\nu)$:

$$e_{0,0} = 1, \qquad e_{1,0} = \pi^{\frac{1}{4}} z, \qquad e_{2,0} = \sqrt{2} z^2,$$

$$e_{1,1} = \sqrt{\frac{2}{\pi - 2}} \left(z\overline{z} - \frac{1}{\sqrt{\pi}} \right), \qquad e_{2,1} = \frac{2\pi^{\frac{1}{4}}}{\sqrt{4 - \pi}} \left(z^2\overline{z} - \frac{\sqrt{\pi}}{2} z \right),$$

$$e_{2,2} = \frac{\sqrt{2}(\pi - 2)}{\sqrt{\pi^2 - 5\pi + 6}} \left(z^2\overline{z}^2 - \frac{\sqrt{\pi}}{\pi - 2} z\overline{z} + \frac{4 - \pi}{2(\pi - 2)} \right).$$

For $e_{p.q}$ with $p < q$:

$$e_{1,0} = \pi^{\frac{1}{4}} \overline{z}, \qquad e_{2,0} = \sqrt{2}\overline{z}^2, \qquad e_{2,1} = \frac{2\pi^{\frac{1}{4}}}{\sqrt{4 - \pi}} \left(\overline{z}^2 z - \frac{\sqrt{\pi}}{2} \overline{z} \right).$$

Operator S

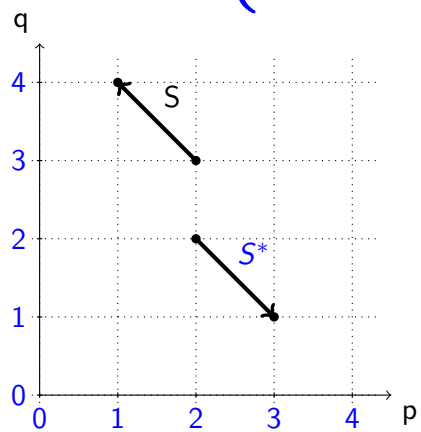
Introduce the operator S on $\mathcal{H}$ defining it on the basis elements

$$Se_{p,q} = \begin{cases} e_{p-1,q+1}, & \text{if} \quad p \geq 1 \\ 0, & \text{if} \quad p = 0 \end{cases}.$$

S is a partial isometry with $\ker S = \widetilde{\mathcal{A}}$ and $(\operatorname{Im} S)^{\perp} = \mathcal{A}$, where $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ are the subspaces of analytic and anti-analytic functions in $\mathcal{H}$.

Its adjoint operator S^* is also a partial isometry with $\ker S^* = \mathcal{A}$ and $(\operatorname{Im} S^*)^{\perp} = \widetilde{\mathcal{A}}$,

$$S^* e_{p,q} = \begin{cases} e_{p+1,q-1}, & \text{if} \quad q \geq 1 \\ 0, & \text{if} \quad q = 0 \end{cases}.$$



Operator S

For each $k \in \mathbb{Z}_+$, we introduce the finite dimensional spaces $\mathcal{H}_k = \text{span}\{e_{p,q} : p+q=k\}$ then, obviously,

$$\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k.$$

Lemma

We have

1. *the following operators are the orthogonal projections* $I - SS^* = P : \mathcal{H} \longrightarrow \mathcal{A}$ *and* $I - S^*S = \widetilde{P} : \mathcal{H} \longrightarrow \widetilde{\mathcal{A}}$,
2. *each subspace* $\mathcal{H}_k$ *in the above direct sum decomposition is invariant under the action of* S *and* S^*,
3. *all the restrictions* $S|_{\mathcal{H}_k}$ *and* $S^*|_{\mathcal{H}_k}$ *are nilpotent,*

$$(S|_{\mathcal{H}_k})^{k+1} = 0 \qquad \text{and} \qquad (S^*|_{\mathcal{H}_k})^{k+1} = 0,$$

Dzuraev '70th, book '92

It was A. Dzuraev who observed first the connection between polyanalytic spaces and the two-dimensional singular integral operators

$$(S_D f)(z) = -\int_D \frac{f(w)\, dA(w)}{(w-z)^2}, \quad (S_D^* f)(z) = -\int_D \frac{f(w)\, dA(w)}{(\overline{w}-\overline{z})^2},$$

where $dA(w) = \frac{1}{\pi} dudv$, with $w = u + iv$.
In particular, he proved that for a bounded domain D with a smooth boundary the projection P_n and $\widetilde{P}_n$ of $L_2(D)$ onto $\mathcal{A}_n$ and onto $\widetilde{\mathcal{A}}_n$ are respectively given by

$$P_n = I - (S_D)^n (S_D^*)^n + K_n \qquad \text{and} \qquad \widetilde{P}_n = I - (S_D^*)^n (S_D)^n + \widetilde{K}_n,$$

where K_n and $\widetilde{K}_n$ are compact operators.
For the unit disk $\mathbb{D}$ case, the compact summands are equal to zero.

Operator $S_{\mathbb{D}}$

Proposition

For $\mathcal{H} = L_2(\mathbb{D}, dA(z))$ the operators S and S^* coincide with two-dimensional singular integral operators $S_{\mathbb{D}}$ and $S^*_{\mathbb{D}}$. And thus

- each subspace $\mathcal{H}_k$ in the direct sum decomposition

$$L_2(\mathbb{D}, dA(z)) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k.$$

is invariant under the action of $S_{\mathbb{D}}$ and $S^*_{\mathbb{D}}$,
- for each $k \in \mathbb{Z}_+$, the restrictions $S_{\mathbb{D}}|_{\mathcal{H}_k}$ and $S^*_{\mathbb{D}}|_{\mathcal{H}_k}$ are nilpotent,

$$(S_{\mathbb{D}}|_{\mathcal{H}_k})^{k+1} = 0 \qquad \text{and} \qquad (S^*_{\mathbb{D}}|_{\mathcal{H}_k})^{k+1} = 0.$$

That is, the operators $S_{\mathbb{D}}$ and $S^*_{\mathbb{D}}$ are nothing but the direct sums of the nilpotent operators on the finite dimensional subspaces.

Back to general operators S and S^*

Lemma

For each $n \in \mathbb{N}$ and all $f \in \mathcal{H}$ we have

$$\begin{aligned} S^n(S^*)^n f &= f - \varphi, \qquad \text{for some} \qquad \varphi \in \mathcal{A}_n, \\ (S^*)^n S^n f &= f - \psi, \qquad \text{for some} \qquad \psi \in \widetilde{\mathcal{A}}_n. \end{aligned}$$

Corollary

The following equalities hold

$$\begin{aligned} \mathcal{A}_n &= \ker(S^*)^n = \overline{\text{span}}\,\{e_{p,q} : \ p \in \mathbb{Z}_+, \quad q = 0, 1, ..., n-1\} \\ \widetilde{\mathcal{A}}_n &= \ker S^n = \overline{\text{span}}\,\{e_{p,q} : \ q \in \mathbb{Z}_+, \quad p = 0, 1, ..., n-1\}. \end{aligned}$$

For each $n \in \mathbb{N}$, the exact analogues of the Dzuraev formulas hold:

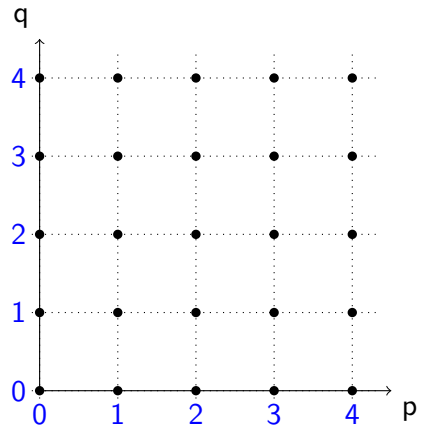
$$P_n = I - S^n(S^*)^n : \mathcal{H} \longrightarrow \mathcal{A}_n \quad \text{and} \quad \widetilde{P}_n = I - (S^*)^n S^n : \mathcal{H} \longrightarrow \widetilde{\mathcal{A}}_n.$$

Pure isometries V and $\widetilde{V}$

We introduce also two pure isometries:

$$V : e_{p,q} \longmapsto e_{p,q+1} \qquad \text{and} \qquad \widetilde{V} : e_{p,q} \longmapsto e_{p+1,q}.$$

For geometric interpretation, we identify each basis element $e_{p,q}$ with the node (p, q) of the lattice $\mathbb{Z}^2_+$.



The isomerty V moves each note one-step up, while the isometry $\widetilde{V}$ moves each note one-step right.

The following relations hold

$$S = V\widetilde{V}^* = \widetilde{V}^* V \qquad \text{and} \qquad S^* = \widetilde{V} V^* = V^* \widetilde{V},$$

Basis oriented descriptions of polyanalytic spaces

We have the following basis oriented descriptions

$$\begin{aligned} \mathcal{A}_n &= \overline{\text{span}}\left\{e_{p,q} : p \in \mathbb{Z}_+, \quad q = 0, 1, \ldots, n-1\right\}, \\ \widetilde{\mathcal{A}}_n &= \overline{\text{span}}\left\{e_{p,q} : q \in \mathbb{Z}_+, \quad p = 0, 1, \ldots, n-1\right\}. \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_{(n)} &= \overline{\text{span}}\left\{e_{p,q} : p \in \mathbb{Z}_+, \quad q = n-1\right\}, \\ \widetilde{\mathcal{A}}_{(n)} &= \overline{\text{span}}\left\{e_{p,q} : q \in \mathbb{Z}_+, \quad p = n-1\right\}. \end{aligned}$$

True-polyanalytic and true-anti-polyanalytic spaces

Lemma

Each true-polyanalytic space $\mathcal{A}_{(n)}$ (true-anti-polyanalytic space $\widetilde{\mathcal{A}}_{(n)}$) is isomorphically isometric to the analytic space $\mathcal{A}$ (to the anti-analytic space $\widetilde{\mathcal{A}}$), and thus all true-polyanalytic spaces (true-anti-polyanalytic spaces) are isomorphically isometric among each other. The corresponding isomorphisms are given by the following operators

$$V^n|_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}_{(n)} \quad \text{and} \quad (V^*)^n|_{\mathcal{A}_{(n)}} : \mathcal{A}_{(n)} \longrightarrow \mathcal{A},$$
$$\widetilde{V}^n|_{\widetilde{\mathcal{A}}} : \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{A}}_{(n)} \quad \text{and} \quad (\widetilde{V}^*)^n|_{\widetilde{\mathcal{A}}_{(n)}} : \widetilde{\mathcal{A}}_{(n)} \longrightarrow \widetilde{\mathcal{A}}.$$

The density of the linear combinations of the monomials $\overline{z}^k z^m$ implies that for each $\mathcal{H}$ in question

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{A}_{(n)} = \bigoplus_{k=1}^{\infty} V^{k-1}(\ker V^*) \quad \text{and} \quad \mathcal{H} = \bigoplus_{n=1}^{\infty} \widetilde{\mathcal{A}}_{(n)} = \cdots.$$

Operators $\mathfrak{a}$ and $\mathfrak{a}^\dagger$

Operators $\mathfrak{a}$, $\mathfrak{a}^\dagger$ and $\widetilde{\mathfrak{a}}$, $\widetilde{\mathfrak{a}}^\dagger$, satisfying $[\mathfrak{a}, \mathfrak{a}^\dagger] = I$, are defined by the following action on basis elements $e_{p,q}$, $p,\ q \in \mathbb{Z}_+$:

$$\mathfrak{a} : e_{p,q} \mapsto \begin{cases} \sqrt{q} e_{p,q-1}, & p \in \mathbb{Z}_+,\ q > 0 \\ 0, & p \in \mathbb{Z}_+,\ q = 0 \end{cases}, \quad \mathfrak{a}^\dagger : e_{p,q} \mapsto \sqrt{q+1} e_{p,q+1},$$

and

$$\widetilde{\mathfrak{a}} : e_{p,q} \mapsto \begin{cases} \sqrt{p} e_{p-1,q}, & q \in \mathbb{Z}_+,\ p > 0 \\ 0, & q \in \mathbb{Z}_+,\ p = 0 \end{cases}, \quad \widetilde{\mathfrak{a}}^\dagger : e_{p,q} \mapsto \sqrt{p+1} e_{p+1,q}.$$

The basis elements of $\mathcal{H}$ have thus the form

$$e_{p,q} = \frac{1}{\sqrt{p!q!}} (\mathfrak{a}^\dagger)^q (\widetilde{\mathfrak{a}}^\dagger)^p e_{0,0} = \frac{1}{\sqrt{p!q!}} (\widetilde{\mathfrak{a}}^\dagger)^p (\mathfrak{a}^\dagger)^q e_{0,0}.$$

Example: weighted Gaussian measure

Let now $\mathcal{H} = L_2(\mathbb{C}, d\mu_\alpha)$, where $d\mu_\alpha = \alpha e^{-\alpha|z|^2} dA(z)$, $\alpha > 0$. Two pairs of mutually adjoint lowering and raising operators

$$\mathfrak{a}_\alpha = \frac{1}{\sqrt{\alpha}} \frac{\partial}{\partial \overline{z}}, \qquad \mathfrak{a}_\alpha^\dagger = -\frac{1}{\sqrt{\alpha}} \frac{\partial}{\partial z} + \sqrt{\alpha}\overline{z},$$

$$\widetilde{\mathfrak{a}}_\alpha = \frac{1}{\sqrt{\alpha}} \frac{\partial}{\partial z}, \qquad \widetilde{\mathfrak{a}}_\alpha^\dagger = -\frac{1}{\sqrt{\alpha}} \frac{\partial}{\partial \overline{z}} + \sqrt{\alpha}z,$$

are defined on the linear span of all monomials $m_{p,q}$, and satisfy the commutation relations $[\mathfrak{a}_\alpha, \mathfrak{a}_\alpha^\dagger] = I$ and $[\widetilde{\mathfrak{a}}_\alpha, \widetilde{\mathfrak{a}}_\alpha^\dagger] = I$.

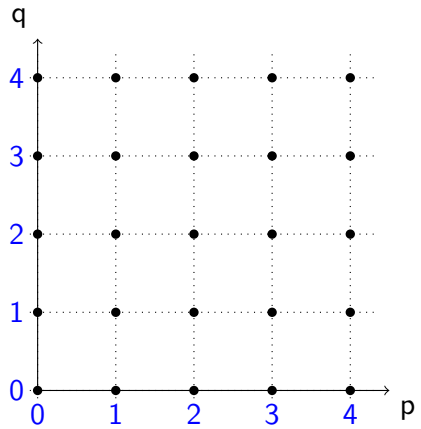
Then the orthonormal basis in $L_2(\mathbb{C}, d\mu_\alpha)$ has the form

$$e_{p,q} = \frac{1}{\sqrt{p!q!}} (\mathfrak{a}_\alpha^\dagger)^q (\widetilde{\mathfrak{a}}_\alpha^\dagger)^p e_{0,0}$$

$$= \sqrt{\alpha^{p+q} p!q!} \sum_{k=0}^{\min\{p,q\}} \frac{(-1)^k}{\alpha^k k!(p-k)!(q-k)!} z^{p-k} \overline{z}^{q-k}, \quad p, q \in \mathbb{Z}_+.$$

Pure isometries $W_{m,n}$

Given any pair $(m,n) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$, we define the pure isometry $W_{m,n} = \widetilde{V}^m V^n = V^n \widetilde{V}^m$.



Geometrically, the isometry $W_{m,n}$ moves each node (p,q) $(\equiv e_{p,q})$ p steps right and q steps up to the node $(p+m, q+n)$ $(\equiv e_{p+m,q+n})$.

Being pure isometry, $W_{m,n}$ implies the direct sum decomposition

$$\mathcal{H} = \bigoplus_{\ell \in \mathbb{Z}_+} W_{m,n}^{\ell}(\ker W_{m,n}^*).$$

Pure isometries $W_{m,n}$

Introduce the subspaces $\mathcal{H}_{(k)}^{(m,n)}$, defined in terms of the pure isometry $W_{m,n}$ and its adjoint $W_{m,n}^*$, as follows:

$$\mathcal{H}_{(k)}^{(m,n)} = W_{m,n}^{k-1}(\ker W_{m,n}^*)$$

Further, we call the space

$$\mathcal{H}_k^{(m,n)} := \ker(W_{m,n}^*)^k = \mathcal{H}_{(1)}^{(m,n)} \oplus \mathcal{H}_{(2)}^{(m,n)} \oplus \ldots \oplus \mathcal{H}_{(k)}^{(m,n)}$$

k-poly-$W_{m,n}$-space, then $\mathcal{H}_{(k)}^{(m,n)} = \mathcal{H}_k^{(m,n)} \ominus \mathcal{H}_{k-1}^{(m,n)}$ is naturally to call true-k-poly-$W_{m,n}$-space

Furthermore,

$$\mathcal{H} = \bigoplus_{k\in\mathbb{N}} \mathcal{H}_{(k)}^{(m,n)}.$$

Characterization of $\mathcal{H}_k^{(m,n)}$

For each pair $(m,n) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$:

- the space $\mathcal{H}^{(m,n)} = \ker W_{m,n}^*$ admits the representation

$$\mathcal{H}^{(m,n)} = \ker S^{m+n}(S^*)^n = \ker (S^*)^{m+n} S^m;$$

- the orthogonal projection $P_{m,n} : \mathcal{H} \longrightarrow \mathcal{H}^{(m,n)}$ has the form

$$P_{m,n} = I - S^n(S^*)^{m+n}S^m = I - (S^*)^m S^{m+n}(S^*)^n.$$

- the k-poly-$W_{m,n}$-space $\mathcal{H}_k^{(m,n)}$ admits the representation

$$\mathcal{H}_k^{(m,n)} = \ker S^{k(m+n)}(S^*)^{kn} = \ker (S^*)^{k(m+n)} S^{km};$$

- the orthogonal projection $P_{m,n;k} : \mathcal{H} \longrightarrow \mathcal{H}_k^{(m,n)}$ has the form

$$P_{m,n;k} = I - S^{kn}(S^*)^{k(m+n)}S^{km} = I - (S^*)^{km} S^{k(m+n)}(S^*)^{kn}.$$

Back to functions

The powers of differential operators $\frac{\partial}{\partial \overline{z}}$ and $\frac{\partial}{\partial z}$ permit us to single out various important subclasses of smooth in a domain $D \subset \mathbb{C}$ functions. A function f is called

- analytic if it satisfies the equation $\frac{\partial}{\partial \overline{z}} f = 0$,
- anti-analytic if it satisfies the equation $\frac{\partial}{\partial z} f = 0$,
- k-polyanalytic if it satisfies the equation $\frac{\partial^k}{\partial \overline{z}^k} f = 0$,
- k-anti-polyanalytic if it satisfies the equation $\frac{\partial^k}{\partial z^k} f = 0$,
- harmonic if it satisfies the equation $\Delta f = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} f = 0$,
- biharmonic if it satisfies the equation $\Delta^2 f = 16 \frac{\partial^2}{\partial z^2} \frac{\partial^2}{\partial \overline{z}^2} f = 0$.

More general, we call a function f (m, n)-analytic if it satisfies the equation $\frac{\partial^m}{\partial z^m} \frac{\partial^n}{\partial \overline{z}^n} f = 0$, and k-(m, n)-polyanalytic if it satisfies the equation $\left(\frac{\partial^m}{\partial z^m} \frac{\partial^n}{\partial \overline{z}^n} \right)^k f = 0$.

(m,n)-analytic and k-(m,n)-polyanalytic functions.

The sum $\varphi+\psi$ of any n-polyanalytic function φ and any m-anti-polyanalytic function ψ is obviously (m,n)-analytic. The converse statement was proven by L. Pessoa, resulting:

Statement

A function f is (m,n)-analytic if and only if it is a sum of n-polyanalytic and m-anti-polyanalytic functions.

Further,

Statement

A function f is k-(m,n)-analytic if and only if it admits the representation

$$f=(z^m\overline{z}^n)^{k-1}g_{k-1}+(z^m\overline{z}^n)^{k-2}g_{k-2}+\ldots+z^m\overline{z}^n g_1+g_0,$$

where all functions g_ℓ are (m,n)-analytic.

(m, n)-analytic function subspaces of $\mathcal{H} = L_2(D, d\nu)$

Denote by $\mathcal{A}^{(m,n)}$ the subspace of $\mathcal{H}$, which consists of all (m, n)-analytic functions.

Examples: The subspaces of analytic, anti-analytic, and harmonic functions are now $\mathcal{A} = \mathcal{A}^{(0,1)}$, $\widetilde{\mathcal{A}} = \mathcal{A}^{(1,0)}$, $\mathsf{H} = \mathcal{A}^{(1,1)}$

For other different valies of (m, n), we have spaces of
- k-polyanalytic functions $\mathcal{A}_k = \mathcal{A}^{(0,k)}$,
- true-k-polyanalytic functions
$\mathcal{A}_{(k)} = \mathcal{A}_k \ominus \mathcal{A}_{k-1} = \mathcal{A}^{(0,k)} \ominus \mathcal{A}^{(0,k-1)}$,
- k-anti-polyanalytic functions $\widetilde{\mathcal{A}}_k = \mathcal{A}^{(k,0)}$,
- true-k-anti-polyanalytic functions
$\widetilde{\mathcal{A}}_{(k)} = \widetilde{\mathcal{A}}_k \ominus \widetilde{\mathcal{A}}_{k-1} = \mathcal{A}^{(k,0)} \ominus \mathcal{A}^{(k-1,0)}$,
- k-polyharmonic functions $\mathsf{H}_k = \mathcal{A}^{(k,k)}$,
- true-k-polyharmonic functions
$\mathsf{H}_{(k)} = \mathsf{H}_k \ominus \mathsf{H}_{k-1} = \mathcal{A}^{(k,k)} \ominus \mathcal{A}^{(k-1,k-1)}$.

Note that all these subspaces are closed in $\mathcal{H} = L_2(D, d\nu)$.

(m, n)-analytic function via isometries $W_{m,n}$

Statement

We have

$$\mathcal{A}_k = \mathcal{H}^{(0,k)} = \ker W^*_{0,k} = \ker (V^*)^k,$$
$$\widetilde{\mathcal{A}}_k = \mathcal{H}^{(k,0)} = \ker W^*_{k,0} = \ker (\widetilde{V}^*)^k,$$
$$\mathrm{H_k} = \mathcal{H}^{(\mathrm{k},\mathrm{k})} = \ker \mathrm{W}^*_{\mathrm{k},\mathrm{k}} = \ker (\mathrm{V}^*)^{\mathrm{k}} (\widetilde{\mathrm{V}}^*)^{\mathrm{k}},$$
$$\mathcal{A}^{(m,n)} = \ker W^*_{m,n} = \ker (V^*)^n (\widetilde{V}^*)^m.$$

Statement

For each predefined analytic quality, $(m, n) \in \mathbb{Z}^2_+ \setminus \{(0, 0)\}$, we have the following orthogonal decomposition of $L_2(D, d\nu)$ onto the true-k-(m, n)-polyanalytic function spaces

$$L_2(D, d\nu) = \bigoplus_{k \in \mathbb{N}} \mathcal{A}^{(m,n)}_{(k)}.$$