Phase space formulation of the Abelian and non-Abelian quantum geometric tensor

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Abstract

We present a formulation of the Berry connection and the quantum geometric tensor in the framework of the phase space or Wigner function formalism. This formulation is obtained through the direct application of the Weyl correspondence to the geometric structure under consideration. In particular, we show that the quantum metric tensor can be computed using only the Wigner functions, which opens an alternative way to experimentally measure the components of this tensor. We also address the non-Abelian generalization and obtain the phase space formulation of the Wilczek-Zee connection and the non-Abelian quantum geometric tensor. In this case, the non-Abelian quantum metric tensor involves only the non-diagonal Wigner functions. Then, we verify our approach with examples and apply it to a system of N coupled harmonic oscillators, showing that the associated Berry connection vanishes and obtaining the analytic expression for the quantum metric tensor.

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Geometry of the parameter space

Consider a quantum system defined by a Hamiltonian $\hat{\mathbf{H}}(\hat{\mathbf{q}}, \hat{\mathbf{p}}; x)$, where $\hat{\mathbf{q}} = \{\hat{\mathbf{q}}_a\}$ and $\hat{\mathbf{p}} = \{\hat{\mathbf{p}}_a\}$ $(a, b, \ldots = 1, \ldots, N)$ are respectively the position and momentum operators, and $x(\tau) = \{x^i\}$ $(i, j, \ldots = 1, \ldots, m)$ is a set of m real adiabatic parameters, i.e., slowly varying functions of time τ.
If $\hat{\mathbf{H}}$ has at least one orthonormal eigenvector $|n(x)\rangle$ with nondegenerate eigenvalue $E_n(x)$, then the Abelian quantum geometric tensor defined in the parameter space of the system is given by

$$Q_{ij}^{(n)} := \langle \partial_i n| \left(1 - |n\rangle\langle n|\right) |\partial_j n\rangle, \tag{1}$$

where $\partial_i = \partial/\partial x^i$.

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By construction, Eq (1) is invariant under the phase transformation

$$|n(x)\rangle \;\to\; |n'(x)\rangle = \mathrm{e}^{i\alpha_n(x)}|n(x)\rangle, \tag{2}$$

where α_n is an arbitrary real function of the parameters.
The (symmetric) real part of Eq. (1) gives the quantum metric tensor [Provost-Valle -1980]

$$g_{ij}^{(n)} = \mathrm{Re}\, Q_{ij}^{(n)}, \tag{3}$$

which defines the line element as $dl^2 = g_{ij}^{(n)}\delta x^i \delta x^j$ and provides a distance between the two neighbor states $|n(x)\rangle$ and $|n(x+\delta x)\rangle$ over the parameter space.

The (antisymmetric) imaginary part of Eq. (1) yields the Berry curvature [Berry1985]

$$F_{ij}^{(n)} = -2\,\mathrm{Im}\,Q_{ij}^{(n)} = \partial_i A_j^{(n)} - \partial_j A_i^{(n)}, \tag{4}$$

which defines the 2-form $F^{(n)} = \frac{1}{2} F_{ij}^{(n)} \delta x^i \wedge \delta x^j$ in the parameter space and is associated with the (Abelian) Berry connection

$$A_i^{(n)} := i \langle n | \partial_i n \rangle. \tag{5}$$

It is worthy of noticing that $A_i^{(n)}(x)$ is real since $\langle n | \partial_i n \rangle$ is purely imaginary and that, under the gauge transformation (2), it changes as

$$A_i^{(n)} \;\rightarrow\; A_i'^{(n)} = A_i^{(n)} - \partial_i \alpha_n, \tag{6}$$

which is the transformation law for a genuine $U(1)$ gauge connection.

Introduction

On the other hand, if the Hamiltonian $\hat{\mathbf{H}}$ has a set of g_n orthonormal eigenvectors $|n_I(x)\rangle$ $(I, J, \ldots = 1, 2, \ldots, g_n)$ associated with the eigenvalue $E_n(x)$, i.e., $\hat{\mathbf{H}}(x)|n_I(x)\rangle = E_n(x)|n_I(x)\rangle$, then the non-Abelian quantum geometric tensor defined in the parameter space of the system is given by [Yu-Quan.2010]

$$Q^{(n)}_{ijIJ} := \langle \partial_i n_I| \left(1 - \sum_{K=1}^{g_n} |n_K\rangle\langle n_K| \right) |\partial_j n_J\rangle. \tag{7}$$

It is easy to see that in the nondegenerate case $g_n = 1$, the non-Abelian tensor (7) reduces to the Abelian one (1).

Furthermore, under the unitary transformation

$$|n_I(x)\rangle \;\to\; |n'_I(x)\rangle = \sum_{J=1}^{g_n} |n_J(x)\rangle U_{JI}(x), \tag{8}$$

where $U_{IJ}(x)$ are the entries of a parameter-dependent unitary $g_n \times g_n$ matrix $U(x)$.

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The tensor (7) transforms covariantly

$$Q_{ijIJ}^{(n)} \rightarrow Q_{ijIJ}^{\prime(n)} = \sum_{K,L=1}^{g_n} U_{KI}^{*} Q_{ijKL}^{(n)} U_{LJ}, \tag{9}$$

where '$*$' stands for complex conjugation.
By analogy with Eqs. (3) and (4), the corresponding (symmetric) non-Abelian quantum metric tensor and the (antisymmetric) non-Abelian Berry curvature or Wilczek-Zee curvature are given by

$$g_{ijIJ}^{(n)} = \frac{1}{2}(Q_{ijIJ}^{(n)} + Q_{ijJI}^{(n)*}), \tag{10}$$

and

$$F_{ijIJ}^{(n)} = i(Q_{ijIJ}^{(n)} - Q_{ijJI}^{(n)*}), \tag{11}$$

respectively.

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Introduction

Moreover, the associated non-Abelian Berry connection or Wilczek-Zee connection is defined by

$$A_{iIJ}^{(n)} := i\langle n_I | \partial_i n_J \rangle. \tag{12}$$

This connection provides the entries of a $g_n \times g_n$ Hermitian matrix for each i and, under the transformation (8), transforms as a proper non-Abelian gauge potential

$$A_{iIJ}^{(n)} \;\rightarrow\; A_{iIJ}^{\prime(n)} = \sum_{K,L=1}^{g_n} U_{KI}^{*} A_{iKL}^{(n)} U_{LJ} + i \sum_{K=1}^{g_n} U_{KI}^{*} \partial_i U_{KJ}. \tag{13}$$

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Wigner-function formalism

Given an operator $\hat{\mathbf{Q}}(\hat{\mathbf{q}}, \hat{\mathbf{p}}; x)$, the Weyl correspondence associates a quantum phase space function $\mathcal{Q}(q, p; x)$ defined as

$$\mathcal{Q}(q,p;x) = \int_{-\infty}^{\infty} \mathrm{d}^N y \, \mathrm{e}^{-\frac{\mathrm{i} p \cdot y}{\hbar}} \langle q + \tfrac{y}{2} | \hat{\mathbf{Q}}(\hat{\mathbf{q}}, \hat{\mathbf{p}}; x) | q - \tfrac{y}{2} \rangle, \tag{14}$$

which is known as the Weyl transform of $\hat{\mathbf{Q}}$.
An important property of the Weyl correspondence is that it allows to write the trace of the product of two operators, $\hat{\mathbf{Q}}$ and $\hat{\mathbf{O}}$, as

$$\mathrm{Tr}(\hat{\mathbf{Q}}\hat{\mathbf{O}}) = \frac{1}{(2\pi\hbar)^N} \int_{-\infty}^{\infty} \mathrm{d}^N q \, \mathrm{d}^N p \, \mathcal{Q}\mathcal{O}, \tag{15}$$

where $\mathcal{Q}$ and $\mathcal{O}$ are the Weyl transform of these operators.

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Wigner-function formalism

The Wigner function $W_n(q,p;x)$, which is the main tool of this formalism, is defined as the function corresponding to the density operator $\hat{\rho}(x)$. More precisely, for a pure state, $\hat{\rho}_n(x) = |n(x)\rangle\langle n(x)|$, it is given by

$$W_n(q,p;x) = \frac{1}{(2\pi\hbar)^N} \int_{-\infty}^{\infty} \mathrm{d}^N y \, \mathrm{e}^{-\frac{\mathrm{i} p \cdot y}{\hbar}} \psi_n(q + \tfrac{y}{2}; x) \psi_n^*(q - \tfrac{y}{2}; x), \quad (16)$$

where $\psi_n(q + \frac{y}{2} y; x) = \langle q + \frac{y}{2} | n(x) \rangle$ and $\psi_n^*(q - \frac{y}{2}; x) = \langle n(x) | q - \frac{y}{2} \rangle$. Thus, the Wigner function provides a phase space representation of the quantum state $|n(x)\rangle$.

From Eq. (15) with $\hat{\mathbf{Q}} \to \hat{\rho}_n$, it is straightforward to see that the expectation value of an operator $\hat{\mathbf{O}}$ can be written as

$$\langle \hat{\mathbf{O}} \rangle_n = \mathrm{Tr}(\hat{\rho}_n \, \hat{\mathbf{O}}) = \int_{-\infty}^{\infty} \mathrm{d}^N q \, \mathrm{d}^N p \, W_n \, \mathcal{O}. \quad (17)$$

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Abelian case

Let us first consider the Abelian Berry connection. We start by writing Eq. (5) as

$$A_i^{(n)} = \mathrm{Tr}\, \hat{\mathbf{A}}_i^{(n)}, \tag{18}$$

where $\hat{\mathbf{A}}_i^{(n)}$ is a quantum operator defined by

$$\hat{\mathbf{A}}_i^{(n)} := i|\partial_i n\rangle\langle n|. \tag{19}$$

Note that $\hat{\mathbf{A}}_i^{(n)}$ is non-Hermitian, and then its Weyl transform

$$\mathcal{A}_i^{(n)}(q,p;x) \;=\; i\int_{-\infty}^{\infty} \mathrm{d}^N y\, \mathrm{e}^{-\frac{\mathrm{i}p\cdot y}{\hbar}}\, \partial_i \psi_n(q+\tfrac{y}{2};x)\, \psi_n^*(q-\tfrac{y}{2};x), \tag{20}$$

is a complex function in phase space variables. Actually, by using Eqs. (16) and (20), it is not hard to show that

$$\mathrm{Im}(\mathcal{A}_i^{(n)}) = \frac{(2\pi\hbar)^N}{2}\partial_i W_n. \tag{21}$$

Abelian case

Having Eq. (18), we can now apply Eq. (15) to relate the phase space function $\mathcal{A}_i^{(n)}$ with the Berry connection. Indeed, taking $\hat{\mathbf{Q}} \to \hat{\mathbb{1}}$ and $\hat{\mathbf{O}} \to \hat{\mathbf{A}}_i^{(n)}$ in Eq. (15) and using Eq. (18), we have

$$A_i^{(n)} = \frac{1}{(2\pi\hbar)^N} \int_{-\infty}^{\infty} \mathrm{d}^N q \, \mathrm{d}^N p \, \mathcal{A}_i^{(n)}. \tag{22}$$

This equation provides an expression of the Berry connection in the Wigner-function formalism. It can be checked that, under the gauge transformation (2), the function $\mathcal{A}_i^{(n)}$ transforms according to

$$\mathcal{A}_i^{(n)} \;\to\; \mathcal{A}'^{(n)}_i = \mathcal{A}_i^{(n)} - (2\pi\hbar)^N W_n \, \partial_i \alpha_n, \tag{23}$$

from which, together with Eq (17), it follows that the connection (22) satisfies the transformation law (6), as expected.

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Abelian case

We observe that the quantum geometric tensor (1) can be written as

$$Q_{ij}^{(n)} = \mathrm{Tr}\,\hat{\mathbf{Q}}_{ij}^{(n)}, \tag{24}$$

where $\hat{\mathbf{Q}}_{ij}^{(n)}$ is an operator defined by

$$\hat{\mathbf{Q}}_{ij}^{(n)} := (\hat{\mathbf{A}}_i^{(n)\dagger} - \hat{\mathbf{A}}_i^{(n)})\hat{\mathbf{A}}_j^{(n)}, \tag{25}$$

with $\hat{\mathbf{A}}_i^{(n)}$ given by Eq. (19). Therefore, noting that $\hat{\mathbf{A}}_i^{(n)\dagger} - \hat{\mathbf{A}}_i^{(n)} = -\partial_i \hat{\rho}_n$, from Eq. (15) with $\hat{\mathbf{Q}} \to -\partial_i \hat{\rho}_n$ and $\hat{\mathbf{O}} \to \hat{\mathbf{A}}_j^{(n)}$ it follows that

$$Q_{ij}^{(n)} = -i \int_{-\infty}^{\infty} \mathrm{d}^N q\, \mathrm{d}^N p\, \partial_i W_n\, \mathcal{A}_j^{(n)}, \tag{26}$$

Abelian case

which, after using Eq. (21), becomes

$$Q_{ij}^{(n)} = -\frac{2i}{(2\pi\hbar)^N}\int_{-\infty}^{\infty} \mathrm{d}^N q\, \mathrm{d}^N p\, \mathrm{Im}(\mathcal{A}_i^{(n)})\, \mathcal{A}_j^{(n)}. \tag{27}$$

This is a formulation of the quantum metric tensor within the phase space formalism. Notice that we only need the knowledge of $\mathcal{A}_i^{(n)}$ in order to compute the quantum geometric tensor (27).

Separating Eq. (27) into its real and imaginary parts, we obtain the expression for Abelian quantum metric tensor

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Abelian case

$$g_{ij}^{(n)} = \frac{(2\pi\hbar)^N}{2} \int_{-\infty}^{\infty} \mathrm{d}^N q \, \mathrm{d}^N p \, \partial_i W_n \, \partial_j W_n, \tag{28}$$

and the expression for the Abelian Berry curvature tensor

$$F_{ij}^{(n)} = \frac{4}{(2\pi\hbar)^N} \int_{-\infty}^{\infty} \mathrm{d}^N q \, \mathrm{d}^N p \, \mathrm{Im}(\mathcal{A}_i^{(n)}) \, \mathrm{Re}(\mathcal{A}_j^{(n)}). \tag{29}$$

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Non-Abelian case

Here, we extend the phase space description of the parameter space to include the non-Abelian geometrical structures. Following the procedure considered for the Abelian case, we begin by expressing the Wilczek-Zee connection as

$$A^{(n)}_{iIJ} = \mathrm{Tr}\, \hat{\mathbf{A}}^{(n)}_{iIJ}, \tag{30}$$

where now the associated quantum operator takes the form

$$\hat{\mathbf{A}}^{(n)}_{iIJ} := i|\partial_i n_J\rangle\langle n_I|. \tag{31}$$

The Weyl transform for this operator is given by

$$\begin{aligned} \mathcal{A}^{(n)}_{iIJ}(q,p;x) &= i\int_{-\infty}^{\infty} \mathrm{d}^N y\, \mathrm{e}^{-\frac{\mathrm{i}p\cdot y}{\hbar}} \langle q + \tfrac{y}{2}|\partial_i n_J\rangle\langle n_I|q - \tfrac{y}{2}\rangle \\ &= i\int_{-\infty}^{\infty} \mathrm{d}^N y\, \mathrm{e}^{-\frac{\mathrm{i}p\cdot y}{\hbar}} \partial_i \psi_{nJ}(q + \tfrac{y}{2};x)\psi^*_{nI}(q - \tfrac{y}{2};x). \end{aligned} \tag{32}$$

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Non-Abelian case

By combining Eq. (30) and Eq. (15) with $\hat{\mathbf{Q}} \to \hat{\mathbb{1}}$ and $\hat{\mathbf{O}} \to \hat{\mathbf{A}}^{(n)}_{iIJ}$, it is direct to see that the expression for the Wilczek-Zee connection in phase space formalism is

$$A^{(n)}_{iIJ} = \frac{1}{(2\pi\hbar)^N} \int_{-\infty}^{\infty} \mathrm{d}^N q \, \mathrm{d}^N p \, \mathcal{A}^{(n)}_{iIJ}, \tag{33}$$

which is the natural generalization of the (Abelian) Berry connection (22).

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Non-Abelian case

Defining the non-diagonal Wigner functions

$$W_{nIJ} := \frac{1}{(2\pi\hbar)^N} \int_{-\infty}^{\infty} \mathrm{d}^N y \, \mathrm{e}^{-\frac{\mathrm{i}p\cdot y}{\hbar}} \psi_{nI}(q + \tfrac{y}{2}; x) \psi^*_{nJ}(q - \tfrac{y}{2}; x). \tag{34}$$

that satisfy

$$\int_{-\infty}^{\infty} \mathrm{d}^N q \, \mathrm{d}^N p \, W_{nIJ} = \delta_{IJ}, \tag{35}$$

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Non-Abelian case

Let us now prove that the connection (33) obeys the gauge transformation law (13) for the Wilczek-Zee connection. Under the unitary transformation (8), the phase space function $\mathcal{A}^{(n)}_{iIJ}$ changes as

$$\mathcal{A}^{(n)}_{iIJ} \;\rightarrow\; \mathcal{A}'^{(n)}_{iIJ} = \sum_{K,L=1}^{g_n} \left[U^*_{KI} \mathcal{A}^{(n)}_{iKL} U_{LJ} + i(2\pi\hbar)^N \, U^*_{KI} \, W_{nLK} \, \partial_i U_{LJ} \right], \quad (36)$$

which generalizes Eq. (23). Note that while the second term on the right side of Eq. (23) is purely imaginary, the corresponding term in Eq. (36) is complex in general. Now, taking into account Eq (36), Eq (33) immediately implies

$$A^{(n)}_{iIJ} \;\rightarrow\; A'^{(n)}_{iIJ} = \sum_{K,L=1}^{g_n} \left[U^*_{KI} A^{(n)}_{iKL} U_{LJ} + i \, U^*_{KI} \partial_i U_{LJ} \int_{-\infty}^{\infty} \mathrm{d}^N q \, \mathrm{d}^N p \, W_{nLK} \right], (37)$$

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which, after using Eq. (35), becomes the transformation law (13).

We now turn to the phase space formulation of the non-Abelian quantum geometric tensor. We find that Eq. (7) can be written as

$$Q_{ijIJ}^{(n)} = \mathrm{Tr}\, \hat{\mathbf{Q}}_{ijIJ}^{(n)}, \tag{38}$$

where the operator $\hat{\mathbf{Q}}_{ij}^{(n)}$ is defined as

$$\hat{\mathbf{Q}}_{ij}^{(n)} := -i\, \partial_i \hat{\mathfrak{P}}_n\, \hat{\mathbf{A}}_{jIJ}^{(n)}. \tag{39}$$

with $\hat{\mathfrak{P}}_n := \sum_{I=1}^{g_n} |n_I\rangle\langle n_I|$ the projection operator. Then, taking $\hat{\mathbf{Q}} \to -\partial_i \hat{\mathfrak{P}}_n$ and $\hat{\mathbf{O}} \to \hat{\mathbf{A}}_{jIJ}^{(n)}$ in Eq. (15), it is clear that Eq (38) is equivalent to

$$Q_{ijIJ}^{(n)} = -i \int_{-\infty}^{\infty} \mathrm{d}^N q\, \mathrm{d}^N p \sum_{K=1}^{g_n} \partial_i W_{nKK}\, \mathcal{A}_{jIJ}^{(n)}, \tag{40}$$

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Non-Abelian case

Let us write this equation directly in terms of the phase space function $\mathcal{A}^{(n)}_{iIJ}$. Using Eqs. (32) and (34), it can be demonstrated that

$$\mathrm{Im}(\mathcal{A}^{(n)}_{iKK}) = \frac{(2\pi\hbar)^N}{2}\partial_i W_{nKK}, \tag{41}$$

which is the analog of Eq. (21). Thus, substituting Eq. (41) into Eq. (40), we get

$$Q^{(n)}_{ijIJ} = \frac{-2i}{(2\pi\hbar)^N}\int_{-\infty}^{\infty} \mathrm{d}^N q\, \mathrm{d}^N p \sum_{K=1}^{g_n} \mathrm{Im}(\mathcal{A}^{(n)}_{iKK})\, \mathcal{A}^{(n)}_{jIJ}, \tag{42}$$

which is an expression for the non-Abelian quantum geometric tensor in the phase space formalism.

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Non-Abelian case

Now we are in a position to find the expressions for the non-Abelian quantum metric tensor and Wilczek-Zee curvature. From Eqs. (10) and (42), the non-Abelian quantum metric tensor turns out to be

$$g_{ijIJ}^{(n)} = \frac{-i}{(2\pi\hbar)^N} \int_{-\infty}^{\infty} \mathrm{d}^N q \, \mathrm{d}^N p \sum_{K=1}^{g_n} \mathrm{Im}(\mathcal{A}_{iKK}^{(n)})(\mathcal{A}_{jIJ}^{(n)} - \mathcal{A}_{jJI}^{(n)*}), \tag{43}$$

which, with the help of (41), takes the form

$$g_{ijIJ}^{(n)} = \frac{(2\pi\hbar)^N}{2} \int_{-\infty}^{\infty} \mathrm{d}^N q \, \mathrm{d}^N p \sum_{K=1}^{g_n} \partial_i W_{nKK} \, \partial_j W_{nIJ}, \tag{44}$$

whereas from Eqs. (11) and (42), the Wilczek-Zee curvature is recasted as

$$F_{ijIJ}^{(n)} = \frac{2}{(2\pi\hbar)^N} \int_{-\infty}^{\infty} \mathrm{d}^N q \, \mathrm{d}^N p \sum_{K=1}^{g_n} \mathrm{Im}(\mathcal{A}_{iKK}^{(n)})(\mathcal{A}_{jIJ}^{(n)} + \mathcal{A}_{jJI}^{(n)*}). \tag{45}$$

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Non-Abelian quantum metric tensor for three coupled oscillators

Let us consider a quantum mechanical system composed of three coupled oscillators and described by the Hamiltonian

$$\hat{\mathbf{H}} = \frac{1}{2}\left\{\sum_{a=1}^{3}\left(\hat{\mathbf{p}}_a^2 + k\hat{\mathbf{q}}_a^2\right) + k'\left[(\hat{\mathbf{q}}_1 - \hat{\mathbf{q}}_2)^2 + (\hat{\mathbf{q}}_2 - \hat{\mathbf{q}}_3)^2 + (\hat{\mathbf{q}}_3 - \hat{\mathbf{q}}_1)^2\right]\right\}, \quad (46)$$

where $x = \{x^i\} = (k, k')$ with $i, j, \ldots = 1, 2$ are the adiabatic parameters. It is convenient to start by introducing the linear transformation

$$\hat{\mathbf{Q}}_a = \sum_{b=1}^{N} U_{ab}\hat{\mathbf{q}}_b, \qquad \hat{\mathbf{P}}_a = \sum_{b=1}^{N} U_{ab}\hat{\mathbf{p}}_b, \quad (47)$$

where U_{ab} are the entries of an $N \times N$ orthogonal matrix U such that $K = U^T\Omega^2 U$, with $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_N)$ being a diagonal matrix whose elements ω_a are the frequencies of the system.

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Non-Abelian quantum metric tensor for three coupled oscillators

Bearing in mind the linear transformation (47) with $N = 3$ and the parameter-independent matrix

$$U = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}, \tag{48}$$

the Hamiltonian (46) can be put in the form

$$\hat{\mathbf{H}} = \frac{1}{2}\sum_{a=1}^{3}\left(\hat{\mathbf{P}}_a^2 + \omega_a^2 \hat{\mathbf{Q}}_a^2\right), \tag{49}$$

with the frequencies $\omega_1 = \sqrt{k}$ and $\omega_2 = \omega_3 = \sqrt{k + 3k'}$.

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Non-Abelian quantum metric tensor for three coupled oscillators

Consequently, the normalized wave functions of the system can be written as

$$\psi_{n_1,n_2,n_3}(q_1, q_2, q_3; x) = \psi_{n_1}(Q_1; x)\psi_{n_2}(Q_2; x)\psi_{n_3}(Q_3; x), \quad (50)$$

where $\psi_{n_a}(Q_a; x)$ is the wave function of the $a-$th uncoupled oscillator with quantum number $n_a = 0, 1, 2, \ldots$. The energy eigenvalues, which depend on three quantum numbers n_1, n_2 and n_3, are then given by

$$E_{n_1,n_2,n_3} = \left(n_1 + \frac{1}{2}\right)\hbar\omega_1 + (n_2 + n_3 + 1)\,\hbar\omega_2. \quad (51)$$

Here, we consider the wave functions $\psi_{0,0,1}$ and $\psi_{0,1,0}$, which have the same energy

$$E_1 := \frac{1}{2}\hbar\omega_1 + 2\hbar\omega_2, \quad (52)$$

and then constitute a degenerate set ($g_1 = 2$).

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Non-Abelian quantum metric tensor for three coupled oscillators

By introducing the notation $\{\psi_{(1)I}\} := (\psi_{0,0,1}, \psi_{0,1,0})$ with $I, J = 1, 2$, the associated non-diagonal Wigner functions are obtained by using Eq. (34), which takes the form

$$W_{(1)IJ} = \frac{1}{(2\pi\hbar)^3} \int_{-\infty}^{\infty} \mathrm{d}^3 y \, \mathrm{e}^{-\frac{\mathrm{i} p \cdot y}{\hbar}} \psi_{(1)I}(q + \tfrac{y}{2}; x) \psi_{(1)J}(q - \tfrac{y}{2}; x), \tag{53}$$

and leads to

$$W_{(1)11} = \frac{1}{(\pi\hbar)^3} (\lambda_3 - 1) \mathrm{e}^{-\frac{\lambda_1 + \lambda_2 + \lambda_3}{2}}, \tag{54a}$$

$$W_{(1)12} = \frac{2}{\pi^3 \hbar^4 \omega_2} (P_2 - i\omega_2 Q_2)(P_3 + i\omega_2 Q_3) \mathrm{e}^{-\frac{\lambda_1 + \lambda_2 + \lambda_3}{2}}, \tag{54b}$$

$$W_{(1)21} = \frac{2}{\pi^3 \hbar^4 \omega_2} (P_2 + i\omega_2 Q_2)(P_3 - i\omega_2 Q_3) \mathrm{e}^{-\frac{\lambda_1 + \lambda_2 + \lambda_3}{2}}, \tag{54c}$$

$$W_{(1)22} = \frac{1}{(\pi\hbar)^3} (\lambda_2 - 1) \mathrm{e}^{-\frac{\lambda_1 + \lambda_2 + \lambda_3}{2}}, \tag{54d}$$

where $\lambda_a := 4H_a/\hbar\omega_a$.

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Non-Abelian quantum metric tensor for three coupled oscillators

Then, plugging the expressions for $W_{(1)IJ}$ into Eq. (44), the components of the non-Abelian quantum metric tensor turn out to be

$$\begin{aligned} g^{(1)}_{ij11}(x) = & \quad g^{(1)}_{ij22}(x) = \frac{1}{32}\begin{pmatrix} \frac{1}{\omega_1^4}+\frac{4}{\omega_2^4} & \frac{12}{\omega_2^4} \\ \frac{12}{\omega_2^4} & \frac{36}{\omega_2^4} \end{pmatrix}, \\ g^{(1)}_{ij12}(x) = & \qquad g^{(1)}_{ij21}(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{55}$$

which are exactly the same that can be obtained directly from Eq. (10). Notice that the metric components $g^{(1)}_{ij11}(x)$ and $g^{(1)}_{ij22}(x)$ diverge at the points of the parameter space where the frequencies of the system approach to zero. Besides, note also that $g^{(1)}_{ij11}(x)$ and $g^{(1)}_{ij22}(x)$ can be thought of as invertible matrices, with determinant given by $\det[g^{(1)}_{ij11}(x)] = \det[g^{(1)}_{ij22}(x)] = 9/256\omega_1^4\omega_2^4$.

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Non-Abelian quantum metric tensor for three coupled oscillators

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Quantum Metric Tensor of a High-Order noncommutative theory

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Abstract

A second objective will be to show that non-commutativity will sometimes arise as the imposition of a gauge condition in theories with first-class constraints. We will consider an example of a high-order theory with constraints that imply a noncommutative theory and we compute the quantum geometric tensor for this system.

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First, we will consider an extension of the Chern-Simons Quantum Mechanics to a second order time derivative theory [Lukierski,(1997)], with an additional harmonic term, the Lagrangian chosen has the form

$$L = \frac{m}{2}\dot{x}_i^2 - \frac{\kappa}{2}x_i^2 + \alpha\epsilon_{ij}\dot{x}_i\ddot{x}_j. \tag{1}$$

In order to study the canonical formalism of this theory we follow the Ostrogradski procedure in this case the generalized canonical momenta are defined by

$$p_i = \frac{\partial L}{\partial \dot{x}_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}_i}\right) \quad \text{and} \quad \pi_i = \frac{\partial L}{\partial \ddot{x}_i}, \tag{2}$$

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High order derivative theory

For the Lagrangian (1), one finds

$$p_i = m\dot{x}_i + 2\alpha\epsilon_{ij}\ddot{x}_j \quad \text{and} \quad \pi_i = -\alpha\epsilon_{ij}\dot{x}_j. \tag{3}$$

For this theory our phase space is defined by $(x_i, \dot{x}_i, p_i, \pi_i)$, i.e. this theory has, in principle, a higher number of degrees of freedom. However, due to the equations (3) the variables of the phase space are not independent, then we have constraints. These constrains are

$$\phi_i = \pi_i + \alpha\epsilon_{ij}\dot{x}_j \approx 0. \tag{4}$$

On the other hand, according to the Ostrogradski formalism the canonical Hamiltonian, is given by

$$H_c = \frac{p_i\dot{x}_i}{2} + \frac{\kappa}{2}x_i^2 + \frac{m}{2\alpha}\epsilon_{ij}\pi_i\dot{x}_j - \frac{\epsilon_{ij}}{2\alpha}\pi_i p_j. \tag{5}$$

We compute the evolution of the constraints using the total Hamiltonian given by

$$H = \frac{p_i \dot{x}_i}{2} + \frac{\kappa}{2} x_i^2 + \frac{m}{2\alpha} \epsilon_{ij} \pi_i \dot{x}_j - \frac{\epsilon_{ij}}{2\alpha} \pi_i p_j + \lambda_i \phi_i. \tag{6}$$

Using this Hamiltonian, the evolution of the constraints results

$$\dot{\phi}_i = \{\phi_i, H\} = \frac{\epsilon_{ij}}{2\alpha} \phi_j + 2\alpha \epsilon_{ik} \lambda_k \approx 0. \tag{7}$$

From the above equation we can determine the Lagrange multipliers

$$\lambda_i \approx -\frac{\phi_i}{4\alpha^2},$$

and in consequence we don't have more constraints. Furthermore, these constraints are second class, with the Poisson bracket given by

$$\{\phi_i, \phi_j\} = 2\alpha \epsilon_{ij}, \tag{8}$$

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High order derivative theory

Now, according of Dirac formalism we have to construct the Dirac brackets, these take the following form

$$\{A, B\}_D = \{A, B\} - \{A, \phi_i\}\{\phi_i, \phi_j\}^{-1}\{\phi_j, B\}. \tag{9}$$

In particular, for our theory we have that the matrix $\{\phi_i, \phi_j\}^{-1}$, is given by

$$\{\phi_i, \phi_j\}^{-1} = -\frac{\epsilon_{ij}}{2\alpha}, \tag{10}$$

Now, by promoting these brackets to commutators we obtain the following algebra between our operators

$$[x_i, p_j] = i\delta_{ij}\mathbb{I}, \quad [x_i, x_j] = 0, \quad [\dot{x}_i, \dot{x}_j] = -\frac{i}{2\alpha}\epsilon_{ij}, \tag{11}$$

From this algebra we see that the variables associated with the velocities are noncommutative.

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Perturbative Approximation

In order to obtain a theory without high order time derivatives in our model and in this way eliminate the states of negative energy. We will use the perturbative method proposed in [Tai-Chung,Nucl.Phys.B2002] The equations of motion for the Lagrangian (1) are

$$\ddot{x}_i = -\frac{\kappa}{m}x_i - \frac{2\alpha}{m}\epsilon_{ij}x_j^{(3)}. \tag{12}$$

Now we assume that the contribution of the high order term is weaker than the other terms in the Lagrangian, consequently we make the assumption that $\alpha << 1$. Then, the second order time derivatives can be approached as

$$\ddot{x}_i \approx -\left(\frac{\kappa}{m} + \frac{4\alpha^2\kappa^2}{m^4}\right)x_i + \left(\frac{2\alpha\kappa}{m^2} + \frac{16\alpha^3\kappa^2}{m^5}\right)\epsilon_{ij}\dot{x}_j + \mathcal{O}(\alpha^4). \tag{13}$$

Higher orders in α, are obtained by iterating the equations of motion.

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Perturbative Approximation

The next step is to built the symplectic form, by using the brackets (11) and the constraints (4) we obtain

$$\Omega = \frac{\omega_{AB}}{2} dz^A \wedge dz^B = \delta_{ij} dp_i \wedge dx_j + \alpha \epsilon_{ij} d\dot{x}_i \wedge d\dot{x}_j. \tag{14}$$

Now, with our approximations the momenta (3) are given to order α^3 as

$$p_i = \left(m - \frac{4\alpha^2\kappa}{m^2}\right)\dot{x}_i - \left(\frac{2\alpha\kappa}{m} + \frac{8\alpha^3\kappa^2}{m^4}\right)\epsilon_{ij}x_j + \mathcal{O}(\alpha^4), \qquad \pi_i = -\alpha\epsilon_{ij}\dot{x}_j. \tag{15}$$

Introducing the above momenta (15) in the symplectic form (14), we obtain

$$\Omega = \left(m - \frac{4\alpha^2\kappa}{m^2}\right)\delta_{ij} d\dot{x}_i \wedge dx_j + \left(\frac{2\alpha\kappa}{m} + \frac{8\alpha^3\kappa^2}{m^4}\right)\epsilon_{ij} dx_i \wedge dx_j + \alpha\epsilon_{ij} d\dot{x}_i \wedge d\dot{x}_j + \ldots \tag{16}$$

This two-form is the approximation to order α^3.

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Perturbative Approximation

By using the symplectic form we read the basic new brackets, $(\omega^{AB})_{ij} = \{z_i, z_j\}_D$ (where $z_i = \{x_1, x_2, \dot{x}_1, \dot{x}_2\}$), explicitly these parenthesis are given by

$$\{x_i, x_j\}_D = \left(\frac{2\alpha}{m^2} + \frac{32\alpha^3\kappa}{m^5}\right)\epsilon_{ij}, \quad \{\dot{x}_i, \dot{x}_j\}_D = \left(\frac{4\alpha\kappa}{m^3} + \frac{80\alpha^3\kappa^2}{m^6}\right)\epsilon_{ij}, \tag{17}$$

$$\{x_i, \dot{x}_j\}_D = \frac{1}{m}\left(1 + \frac{12\alpha^2\kappa}{m^3}\right)\delta_{ij}. \tag{18}$$

To avoid the additional extra constant factor in (18), we define

$$\rho_i = (1 - \frac{12\alpha^2\kappa}{m^3})m\dot{x}_i, \tag{19}$$

in consequence the basic parenthesis in this case are

$$\{x_i, x_j\}_D = \left(\frac{2\alpha}{m^2} + \frac{32\alpha^3\kappa}{m^5}\right)\epsilon_{ij}, \quad \{\rho_i, \rho_j\}_D = \left(\frac{4\alpha\kappa}{m} - \frac{16\alpha^3\kappa^2}{m^4}\right)\epsilon_{ij}, \tag{20}$$

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On the other hand, if we introduce the momenta (15) in the Hamiltonian (5) and the definition of ρ_i, we obtain the Hamiltonian in terms of the new variables. So, to third order in α we get

$$H = \frac{1}{2m}\left(1 + \frac{16\alpha^2\kappa}{m^3}\right)\rho_i^2 + \frac{\kappa}{2}x_i^2 + \left(\frac{2\alpha\kappa}{m^2} + \frac{32\alpha^3\kappa^2}{m^5}\right)\epsilon_{ij}x_i\rho_j + \mathcal{O}(\alpha^4). \quad (21)$$

In this way, directly from the high order theory we get a noncommutative theory with Dirac brackets in the reduced phase space given by (20) and Hamiltonian (21).

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Cuantization

Now, the more simple way to quantize this noncommutative theory is to map noncommutative phase space to the ordinary phase space, which satisfy the following commutation relations

$$\{\bar{x}_i, \bar{x}_j\} = \{\bar{\rho}_i, \bar{\rho}_j\} = 0, \quad \{\bar{x}_i, \bar{\rho}_j\} = \delta_{ij}. \tag{22}$$

The mapping that relates the new variables to the old variables is given by

$$x_i = A_{ij}\bar{x}_j + B_{ij}\bar{\rho}_j, \quad \rho_i = C_{ij}\bar{x}_j + D_{ij}\bar{\rho}_j, \tag{23}$$

In the which **A**, **B**, **C** and **D** are $\mathbf{2} \times \mathbf{2}$ transformation matrices.

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Cuantization

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Therefore, the transformations take the following form

$$x_i = a\bar{x}_i - \frac{1}{a}\left(\frac{\alpha}{m^2} + \frac{16\alpha^3\kappa}{m^5}\right)\epsilon_{ij}\bar{\rho}_j + \dots, \quad (24)$$

$$\rho_i = \frac{1}{a}\left(1 - \frac{2\alpha^2\kappa}{am^3}\right)\bar{\rho}_i + a\left(\frac{2\alpha\kappa}{m} - \frac{4\alpha^3\kappa^2}{m^4}\right)\epsilon_{ij}\bar{x}_j + \dots \quad (25)$$

These transformations (22) will allow us to quantized our theory. Introducing the transformations (24) in the Hamiltonian (21), this takes the form

$$H = \mathrm{A}^2(\alpha,\kappa,m)\bar{\rho}_i^2 + \mathrm{B}^2(\alpha,\kappa,m)\bar{x}_i^2 + \mathrm{C}(\alpha,\kappa,m)\epsilon_{ij}\bar{x}_i\bar{\rho}_j, \quad (26)$$

where, the constants parameters $(\mathrm{A},\mathrm{B},\mathrm{C})$, up to third order in α, are

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Cuantization

$$\mathrm{A}^2(\alpha,\kappa,m) = \frac{1}{2ma^2}\left(1+\frac{9\alpha^2\kappa}{m^4}+\mathcal{O}(\alpha^4)\right),$$

$$\mathrm{B}^2(\alpha,\kappa,m) = a^2\left(\frac{\kappa}{2}-\frac{2\alpha^2\kappa^2}{m^3}+\mathcal{O}(\alpha^4)\right),$$

$$\mathrm{C}(\alpha,\kappa,m) = -\frac{\alpha\kappa}{m^2}-\frac{8\alpha^3\kappa^2}{m^5}+\mathcal{O}(\alpha^4). \tag{27}$$

We recognize (26) as the Hamiltonian for the commutative, isotropic 2-dimensional harmonic oscillator, with a coupling term proportional to the L_z angular momentum.

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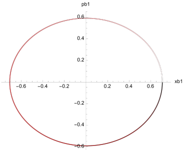
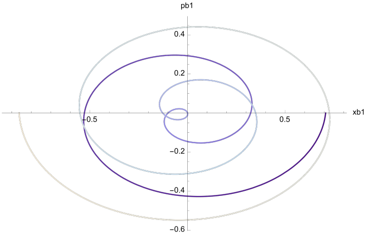
Phase space plots



(a) $\alpha = 0, \; k = 0{,}7$



(b) $\alpha = 0{,}1$ and $k = 0{,}5$

Figura 1: Phase space plots for the coordinates $\bar{x}_1$ and $\bar{\rho}_1$

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Cuantization

To quantize the theory we use the phase space formalism of Quantum Mechanics with the star value equation

$$H^W \star W_{n_1,n_2}(\bar{x},\bar{\rho}) = E_{n_1,n_2} W_{n_1,n_2}(\bar{x},\bar{\rho}) \tag{28}$$

with the solution [Bernardini-Bertolami.2013]

$$W_{n_1,n_2}(\bar{x},\bar{\rho}) = \frac{(-1)^{n_1+n_2}}{\pi^2} \exp\left[-\left(\frac{A}{B}\bar{x}^2 + \frac{B}{A}\bar{\rho}^2\right)\right] L^0_{n_1}(\Omega_+) L^0_{n_2}(\Omega_-) \tag{29}$$

where L^0_n are the associated Laguerre polynomials, n_1 and n_2 are non-negative integers, and

$$\Omega_\pm = \frac{A}{B}\bar{x}^2 + \frac{B}{A}\bar{\rho}^2 \mp 2\epsilon_{ij}\bar{\rho}_i\bar{x}_j \tag{30}$$

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with the energy spectrum

$$E_{n_r,l} = 2\sqrt{AB}(n_1+n_2+1)+C(n_1-n_2) = 2\sqrt{AB}(2n_r+|l|+1)+lC \quad (31)$$

with quantum numbers taking values $n_r = 0, 1, 2, \ldots$, $l = 0, \pm1, \pm2, \ldots$. Note that the spectrum only depends of the constant a, the mass of the system and the parameters κ, α, so that the spectrum of the system is uniquely determined.

Also, this spectrum has a well-defined minimum energy state. In order to make clear this fact, we define the following positive numbers (n_1, n_2), which are determined as follows

$$n_r = n_2 + \frac{l - |l|}{2}, \qquad l = n_1 - n_2. \tag{32}$$

Introducing these quantum numbers in the energy (31), we obtain

$$E_{n_1,n_2} = \sqrt{\frac{\kappa}{m}}\left[1+\frac{5\alpha^2\kappa}{2m^3}+\mathcal{O}(\alpha^4)\right](n_1+n_2+1)+\left[\frac{\alpha\kappa}{m^2}+\frac{8\alpha^3\kappa^2}{m^5}+\mathcal{O}(\alpha^5)\right](n_1-n_2 \tag{33}$$

Therefore, for minimum energy state we get

$$E_{0,0} = \sqrt{\frac{\kappa}{m}}\left[1 + \frac{5\alpha^2\kappa}{2m^3} + \mathcal{O}(\alpha^4)\right], \tag{34}$$

this energy is positive definite. We can also see that in the limit $\alpha \to 0$ we recover the usual case of two harmonic oscillators.

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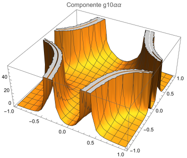
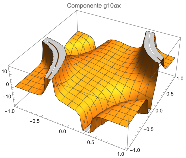
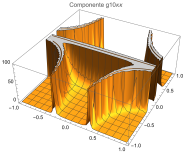
Quantum Metric Tensor



(a) Quantum Metric tensor component $g_{\alpha\alpha}$



(b) Quantum Metric tensor component $g_{\alpha k}$



(c) Quantum Metric tensor component g_{kk}

Figura 2: Components of the Quantum Metric Tensor

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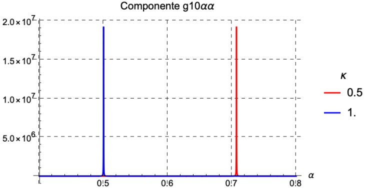
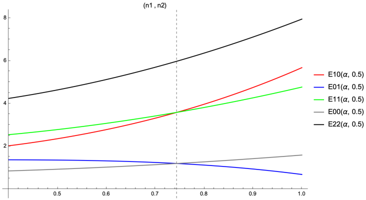
Quantum Metric Tensor



(a) Two dimensional map of the Quantum Metric tensor component $g_{\alpha\alpha}$, with $k = 0{,}5$ and $k = 1$



(b) Energy levels for $k = 0{,}5$

Figura 3: Two dimensional map of $g_{\alpha\alpha}$, and energy levels for $k = 0{,}5$

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Gracias

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