Cruzamientos Evitados y Entropía de Wehrl en el modelo de Lipkin-Meshkov-Glick

6a reunión: Caos y Termalización

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CONTENTS

Research questions:

1. What is the behaviour of Werhl entropy in the vicinity of Avoided Crossings?
2. How this behaviour varies when J takes larger values?

INTRODUCTION

INTRODUCTION

The Lipkin-Meshkov-Glick model is a many-body system that can be studied in a simple form because it can be reduced to a one degree of freedom system.

Key concepts:

- In certain regions of EDoS there are ESQPT.
- This model exhibit Avoided Crossings
- Dynamical tunneling.
  - Superposition of the Husimi function.
  - Increase of Wehrl entropy around AC.

The aim of this work is to measure delocalization of the Husimi function through the computation of Wehrl entropy around Avoided Crossings in the Lipkin-Meshkov-Glick model.

THEORETICAL FRAMEWORK

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In terms of pseudospin operators, LMG Hamiltonian is given by

$$\hat{H}_{LMG} = \epsilon_0 \left[\hat{J}_z + \left(\frac{\gamma_x}{2J-1} \right) \hat{J}_x^2 + \left(\frac{\gamma_y}{2J-1} \right) \hat{J}_y^2 \right] \tag{1}$$

Since the Hamiltonian commutes with $\hat{J}^2$, it can be diagonalized in the basis $|J,\, m\rangle$ using the following equations.

$$\hat{J}_z\, |J,\, m\rangle = \hbar m\, |J,\, m\rangle \tag{2}$$

$$\hat{J}_+\, |J,\, m\rangle = \hbar\sqrt{J(J+1) - m(m+1)}\, |J,\, m+1\rangle \tag{3}$$

$$\hat{J}_-\, |J,\, m\rangle = \hbar\sqrt{J(J+1) - m(m-1)}\, |J,\, m-1\rangle \tag{4}$$

- The eigenvalues will depend on coupling parameters γ_x and γ_y.

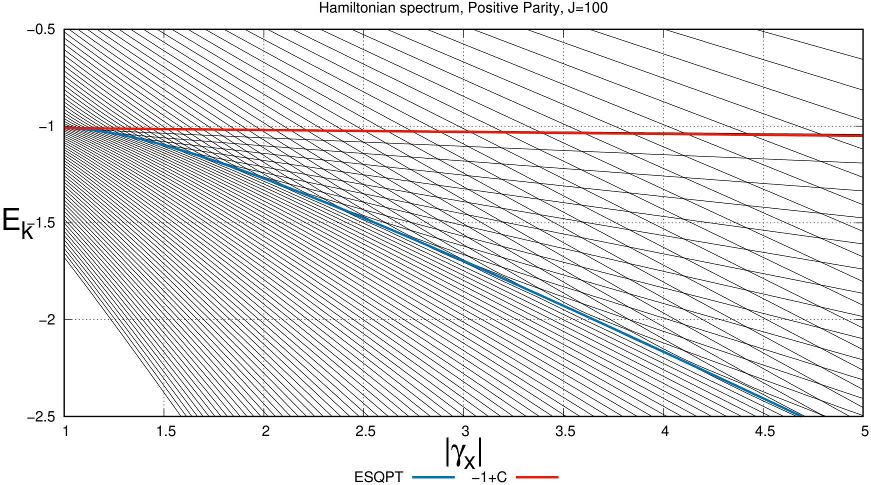


Figure: The Hamiltonian spectrum exhibit Avoided Crossings in the region enclosed by transitions.

- Parameter space according to different behaviour in EDoS.

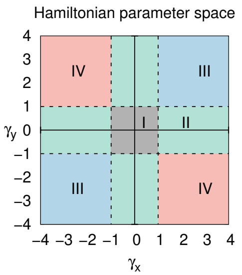
- Semiclassically, the Energy Density of States (EDoS) is given by

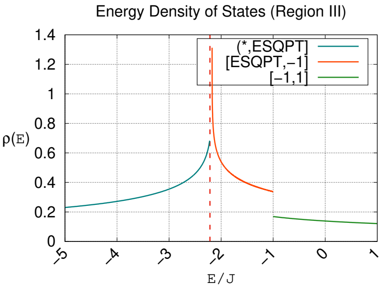
$$\rho(E) = \frac{J}{2\pi} \int \mathrm{d}z\mathrm{d}\phi\, \delta[H(z,\phi) - E] \quad (5)$$

where $H(z,\phi)$ is the semiclassical hamiltonian.

- EDoS exhibits a logarithmic divergence. This defines what is called Excited State Quantum Phase Transition (ESQPT).

- This work focuses on region III.





COHERENT STATE OF THE BLOCH SPHERE

The coherent state of Bloch sphere is useful to analyse Quantum-classical correspondence of classical trajectories and the Husimi function.

These states are constructed applying a rotation operator to minimal projection state $|J,-J\rangle$

$$|\alpha\rangle = \frac{e^{\alpha \hat{J}_+}}{\left(1+|\alpha|^2\right)^J}|J,-J\rangle = \frac{1}{\left(1+|\alpha|^2\right)^J}\sum_{m=-J}^{J}\binom{2J}{J+m}^{1/2}\alpha^{J+m}|J,m\rangle \tag{6}$$

where $\alpha = \tan(\theta/2)e^{-i\phi}$ is a complex number in spherical coordinates.

In general, the eigenstates of the LMG Hamiltonian are given by

$$|E_k\rangle = \sum_{m=-J}^{J} C_m^k |J,\, m\rangle \tag{7}$$

HUSIMI FUNCTION

Denoted by $Q(\alpha)$, is a quasi-probability distribution

1. Always positive.
2. To integrate respect to θ or ϕ does not lead to marginal probability distribution.

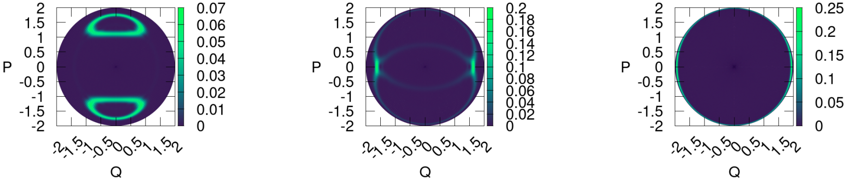
It is defined as the expectation value of a density matrix of the form $\hat{\rho} = |E_k\rangle \langle E_k|$.

$$\boxed{Q_k(\alpha) = |\langle \alpha | E_k \rangle|^2} \tag{8}$$

It is also convenient to define the canonical variables Q and P

$$Q = \sqrt{2(1-\cos\theta)}\cos\phi \quad ; \quad P = -\sqrt{2(1-\cos\theta)}\sin\phi \tag{9}$$

For a fixed value of J, there are $J+1$ eigenvalues for the positive parity. These are sorted in ascending order from $k = 1, \ldots, J+1$.



(a) $k = 15$ (b) $k = 60$ (c) $k = 80$

Figure: Husimi function of eigenstates of the positive parity for a fixed value of $J = 100$, $\gamma_x = -4.0$, $\gamma_y = 3\gamma_x$.

AVOIDED CROSSINGS

The coupling parameters where Avoided Crossings (AC) and Real Crossings take place can be determined through the following condition,

$$\gamma_x \gamma_y = \left(\frac{2J-1}{2J-N} \right)^2 \tag{10}$$

determined from the Einstein-Brillouin-Keller rule.

- $N \in \mathbb{Z}$
- $0 < N < 2J$.
- N_{even} leads to ACs.

Particularly, in this work, $\gamma_y = 3\gamma_x$, then ACs are determined using

$$\left| \gamma_x^{AC} \right| = \frac{1}{\sqrt{3}} \left(\frac{2J-1}{2J-N_{\text{even}}} \right) \tag{11}$$

DYNAMICAL TUNNELING

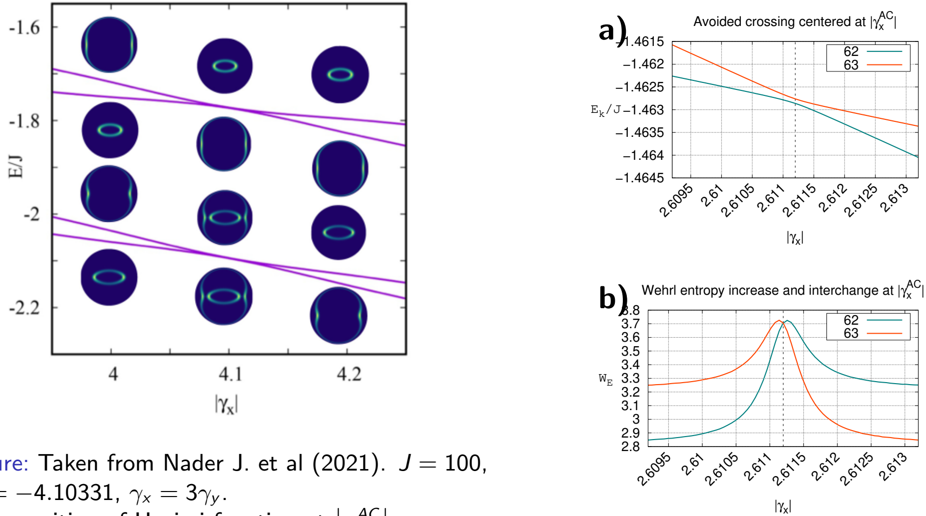


Figure: Taken from Nader J. et al (2021). $J = 100$, $\gamma_x = -4.10331$, $\gamma_x = 3\gamma_y$.
Superposition of Husimi function at $\left|\gamma_x^{AC}\right|$.

ENTROPY AND MONTE CARLO INTEGRATION

Wehrl entropy is used as a delocalization measure of the Husimi function, i.e. the lack of information of where a particular eigenstate is localized in phase space.

In terms of $Q_k(\alpha)$, Wehrl entropy is given by

$$W_E = -\int Q_k(\alpha) \ln Q_k(\alpha) \mathrm{d}\Omega \tag{12}$$

where $\alpha = \tan(\theta/2)e^{-i\phi}$ and $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ is the solid angle of the unitary Bloch sphere.

MONTE CARLO INTEGRATION

An essential part of the entropy computation consists of evaluating the Husimi function and integrating numerically.

Considering a large sample of n points (θ_i, ϕ_i) where $\theta_i \in [0, 2\pi]$ and $\phi_i \in [0, \pi]$. Wehrl entropy can be approximated numerically as follows

$$\begin{aligned} W_{E_k} &= -\int_0^{2\pi}\int_0^{\pi} Q_k(\alpha)_k \ln\left(Q_k(\alpha)\right)\sin\theta \mathrm{d}\theta\mathrm{d}\phi \\ &\approx \left(\frac{2\pi^2}{n}\right)\left(\frac{2J+1}{4\pi}\right)\sum_{i=1}^{n} Q_k(\theta_i,\phi_i)\ln\left(Q_k(\theta_i,\phi_i)\right)\sin\theta_i \end{aligned} \tag{13}$$

where a normalization factor $\left(\frac{2J+1}{4\pi}\right)$ has been introduced.

METHODOLOGY

DIAGONALIZATION

Wolfram Mathematica built-in functions were chosen:

- `Eigenvalues[]`
- `Eigenvectors[]`

How varies the energy difference of consecutive energy levels?

1. Diagonalization for J=100 at γ_x^{AC}
2. 101 energy levels.
3. Sort in ascending order.
4. 50 pairs.

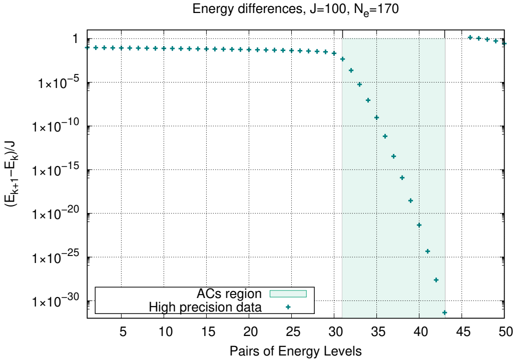


Figure: The green region shows an exponential decrease in the energy gap of avoided crossings.

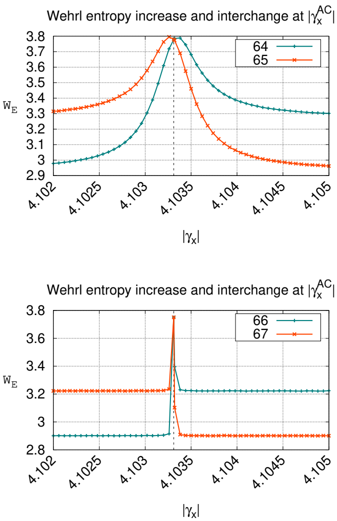


Figure: Wehrl entropy of two consecutive pairs with the same interval of γ_x

- Each pair of states of greater energy require a different interval of γ_x because the interval is significantly smaller.
- What is the rate of decrease in the width of these curves?

Case 1:

- $J = 100$ fixed and different ACs

Case 2:

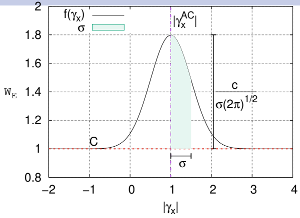
- $J = 100, 200, 500$ and approximately the same γ_x^{AC}

GAUSSIAN FIT

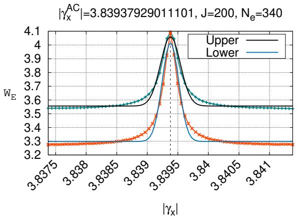
Consider the next Gaussian function

$$f(\gamma_x) = C + \frac{c}{\sigma\sqrt{2\pi}} \exp\left\{ -\frac{(\gamma_x - |\gamma_x^{AC}|)^2}{2\sigma^2} \right\}, \tag{14}$$

- C is the height repect to zero of eq. (14).
- c is part of the factor $\frac{c}{\sigma\sqrt{2\pi}}$ that measures the amplitude of eq. (14).
- σ control the width of eq. (14).

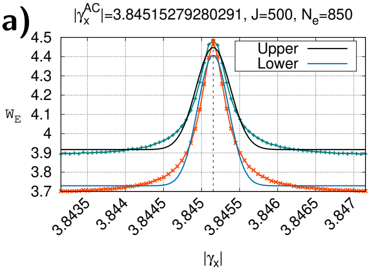


(a) Visualization of parameters.



(b) Gaussian fit example.

PARAMETER $\bar{\sigma}$ OF GAUSSIAN FITS



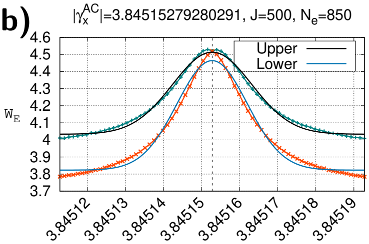


Figure: a) $k = 303 - 304$, b) $305 - 306$

To measure the rate of decreasing in the width, consider

$$\bar{\sigma} \equiv \frac{\sigma_{\text{upper}} + \sigma_{\text{lower}}}{2} \tag{15}$$

An exponential decrease in the width is observed as we take pairs far from ESQPT. Then,

$$\bar{\sigma} = A e^{-\alpha(E_{\text{mean}} - \text{ESQPT})/J} \tag{16}$$

RESULTS

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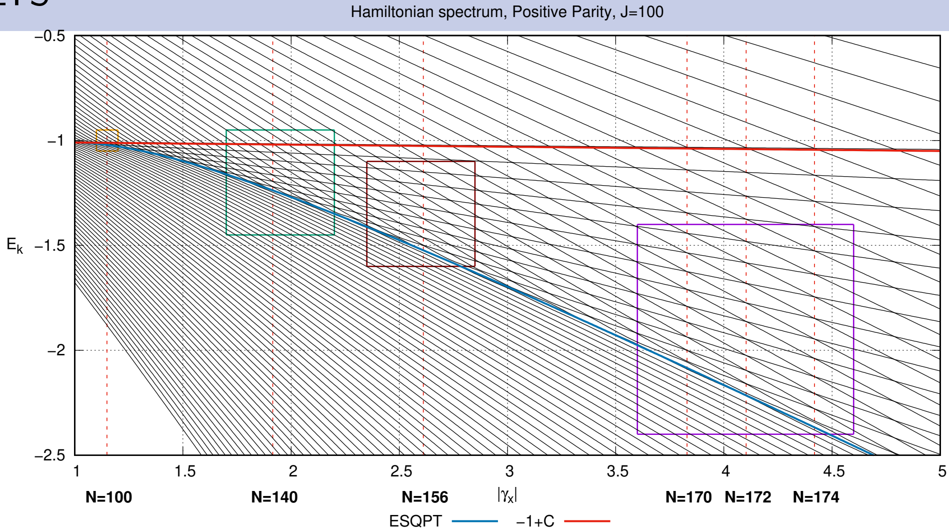
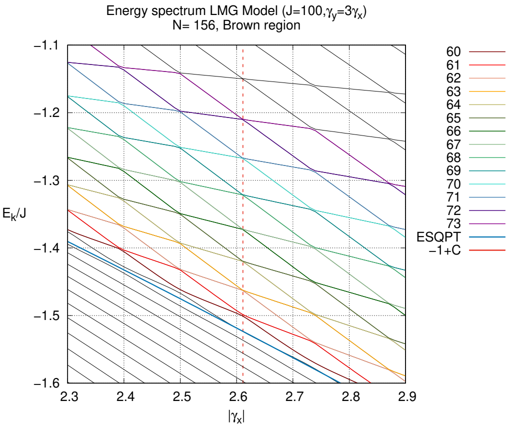


Figure: **Case 1:** $J = 100$ and different ACs.



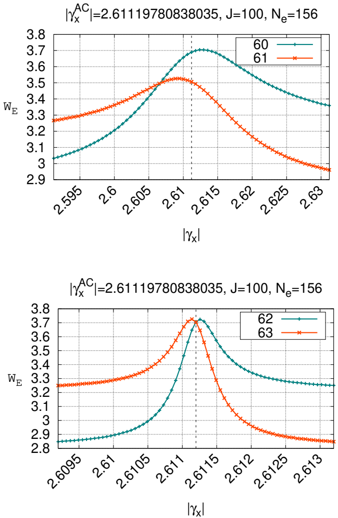
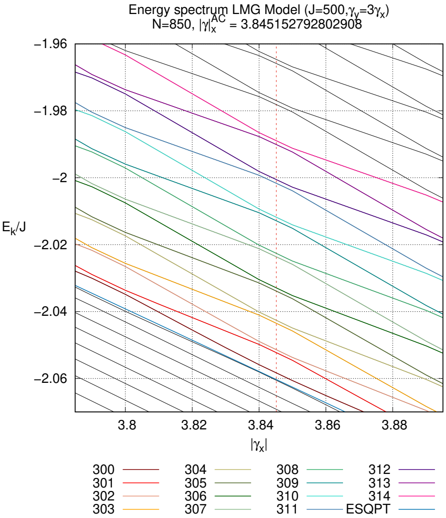


Figure: Wehrl entropy of the first two pairs of states after ESQPT



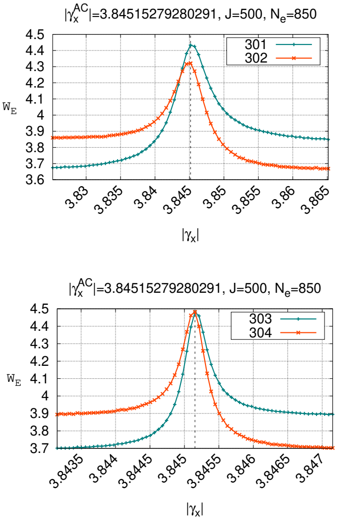


Figure: Wehrl entropy of the first two pairs of states after ESQPT. **Case 2**

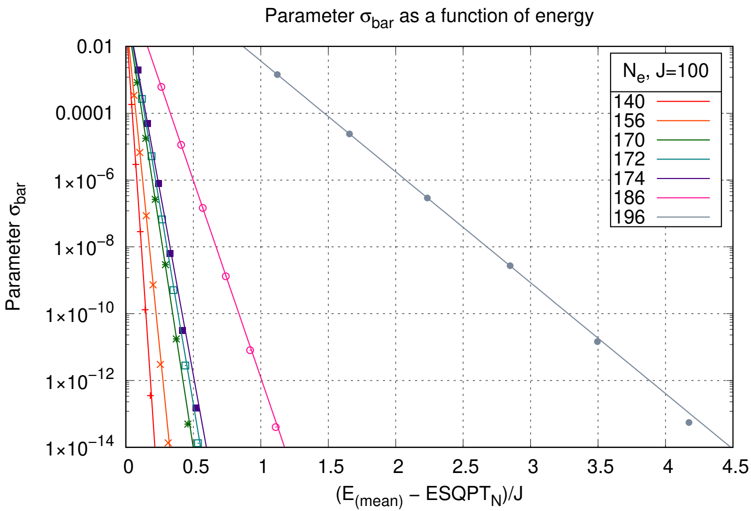


Figure: Parameter $\bar{\sigma}$ in logarithmic scale as a function of $(E_{\text{mean}} - \text{ESQPT})/J$ for different avoided crossings. $\bar{\sigma}$ decreases exponentially.

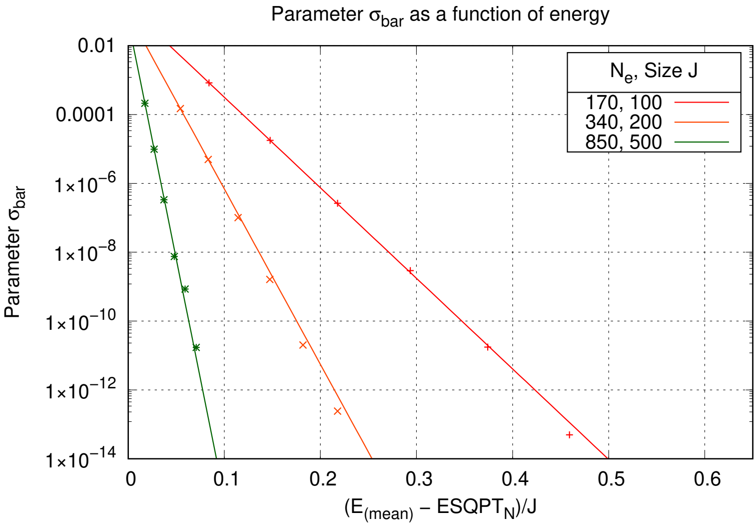


Figure: $\bar{\sigma}$ decreases exponentially.

ENERGY GAP ΔE AT AC

As we observed previously, the energy gap ΔE of consecutive pairs at AC also decreases exponentially. The solid lines in the figure are described by

$$\frac{\Delta E}{J} = Be^{-\beta(E_{\text{mean}} - \text{ESQPT})/J} \tag{17}$$

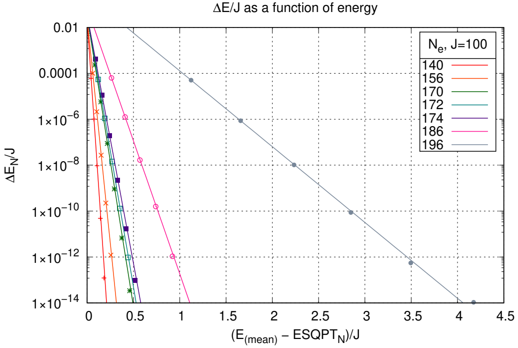


Figure: Energy gap has a similar behaviour along the energy spectrum.

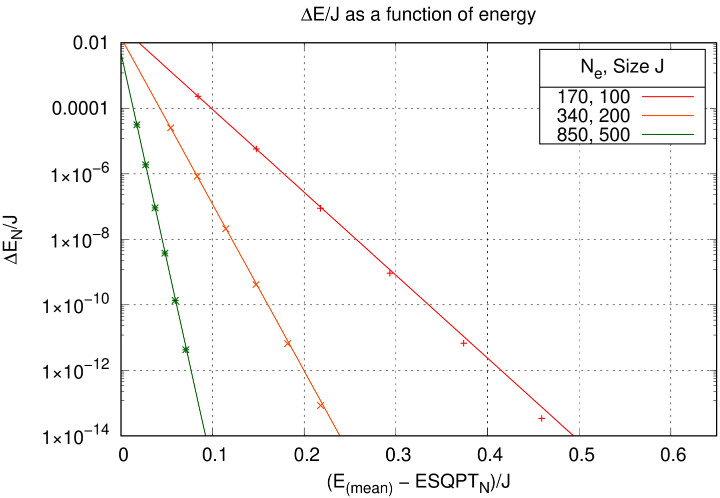


Figure: The range of energy where it is possible to detect Dynnamical Tunneling is narrower as the size of the system increases.

$\bar{\sigma}$ AND ΔE CORRELATION

From expressions for $\bar{\sigma}$ and $\bar{\sigma}$, it is possible to show that

$$\ln \bar{\sigma} = m \ln (\Delta E / J) + b \tag{18}$$

where $m \equiv \frac{\alpha}{\beta}$ and $b \equiv \ln \frac{A}{B^{\alpha/\beta}}$.

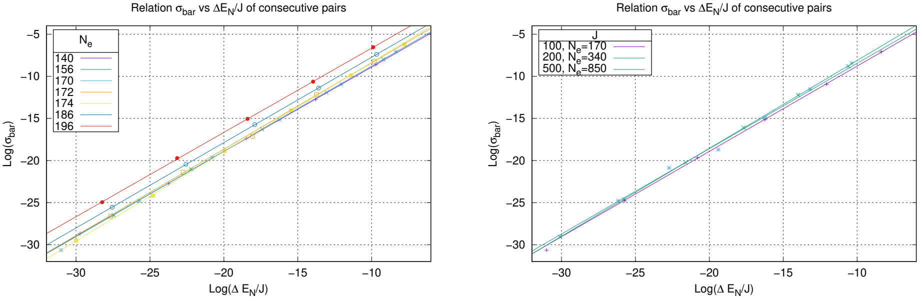


Figure: $\ln \bar{\sigma}$ as a function of $\ln \Delta E$ in both cases. Positive correlation

NUMERICAL RESULTS

Equation (18) implies a power law between $\bar{\sigma}$ and ΔE

$$\bar{\sigma} = C(\Delta E/J)^{\alpha/\beta} \tag{19}$$

with $C = \frac{A}{B^{\alpha/\beta}}$

J	N_e	m	Δm	b	Δb
100	140	1.00398	0.00371	1.1408	0.06019
100	156	1.0109	0.005189	1.32628	0.08536
100	170	1.01285	0.006995	1.33982	0.1119
100	172	1.02573	0.01103	1.86144	0.202
100	174	1.06102	0.007393	2.2779	0.1632
100	186	1.01349	0.001369	2.39909	0.02395
100	196	1.00094	0.003417	3.35955	0.05828
200	340	1.04369	0.004264	2.31694	0.0912
500	850	1.01315	0.0237	1.67272	0.415

Table: Linear fits $\ln \bar{\sigma} = m \ln(\Delta E/J) + b$. The third column shows that $m = \frac{\alpha}{\beta} \approx 1$

CONCLUSIONS

In this project, the width of Wehrl entropy curves was determined for different ACs for a fixed value of J and different system sizes.

Observations:

- The parameter $\bar{\sigma}$ shows an exponential decrease.
- The Energy gap ΔE shows a similar behaviour.

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REFERENCES:

- Lipkin, H. J., Meshkov, N. y Glick, A. J. (1965). Validity of Many-Body Aproximation Methods for a Solvable Model: (I) Exact solutions and perturbation theory. Nuclear Physics 62, 188-198.
- Lerma-H., S. y Dukelsky, J. (2014). The Lipkn-Meshkov-Glick model from the perspective of the SU(1,1) Richardson-Gaudin models. Journal of Physics: Conference Series 492, 012013. DOI: 10 . 1088 / 1742 - 6596 / 492 / 1 / 012013. URL: https://doi.org/10.1088/1742-6596/492/1/012013.
- González-Rodríguez, C. A. (2015). Correspondencia Clásico-Cuántica En el Modelo de Lipkin-Meshkov-Glick. Universidad Veracruzana. 20-31.
- Nader, D. J., González-Rodríguez C. A. y Lerma-Hernández, S. (2021). Avoided Crossings and dynamical tunneling close to excited state quantum phase transitions. Physical Review E. 104, 064116. DOI: 10 . 1103 / PhysRevE . 104 . 064116. URL: https://link.aps.org/doi/10.1103PhysRevE.104.064116.

¡Gracias por su atención!