Una mirada a los sistemas átomo-campo

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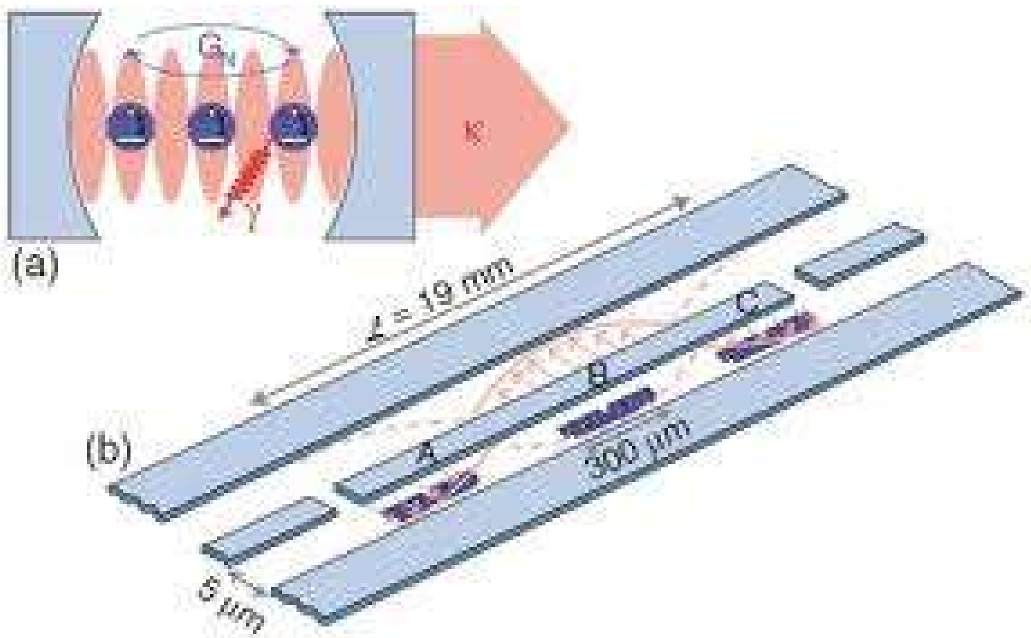
The Dicke Model

- The simplest quantum system with atoms interacting with photons: N two level atoms inside a perfectly reflecting cavity which allows only one electromagnetic mode.

- It also describes superconducting circuits which behave as artificial atoms coupled to a resonator.



R. H. Dicke, Phys. Rev. 93, 99 (1954)

Superradiance: 2 different concepts

Superradiance is a phenomenon that occurs when a group of *N* emitters, such as excited atoms, interact with a common light field. If the wavelength of the light is much greater than the separation of the emitters, then the emitters interact with the light in a collective and coherent fashion.

Gross, M.; Haroche, S. "Superradiance: An essay on the theory of collective spontaneous emission". Physics Reports. ***93*** *(1982) 301–396*

Emmitted power : Independent atoms ~ N, coherent emission in a cavity ~ N^2

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A **superradiant phase transition** is a phase transition that occurs in a collection of fluorescent emitters (such as atoms), between a state containing few electromagnetic excitations (as in the electromagnetic vacuum) and a superradiant state with many electromagnetic excitations trapped inside the emitters.

Atoms interacting with the electromagnetic modes in a cavity

The atoms are in a region smaller than the EM typical wavelength: the spatial variations of the EM field are neglected.

The atoms form a diluted gas: their interactions are neglected.

Two Hamiltonian descriptions, connected via a gauge transformation

$$H_M = \sum_\alpha \left(\frac{\vec{\pi}_\alpha^2}{2m} + V(\vec{r}_\alpha) \right) - e \sum_\alpha \vec{E} \cdot \vec{r}_\alpha + \frac{1}{8\pi} \int \left(\vec{E}^2 + \vec{B}^2 \right) d^3\vec{r}$$

Dipole interaction

$$H_R = \sum_\alpha \left(\frac{\vec{p}_\alpha^2}{2m} + V(\vec{r}_\alpha) \right) - \frac{e}{mc} \sum_\alpha \vec{A} \cdot \vec{p}_\alpha + \frac{e^2}{2mc^2} \vec{A}^2 + \frac{1}{8\pi} \int \left(\vec{E}^2 + \vec{B}^2 \right) d^3\vec{r}$$

Radiation gauge

Diamagnetic term

The atoms are approximated by two-level systems $\frac{\vec{p}_\alpha^2}{2m} + V(\vec{r}_\alpha) \Rightarrow \sigma_{z\alpha}$

$$H_{D,A^2} = \omega a^\dagger a + \omega_0 J_z + \frac{\gamma}{\sqrt{\mathcal{N}}} \left(a + a^\dagger\right) \left(J_+ + J_-\right) + \kappa \left(a + a^\dagger\right)^2$$

The quantum version

The diamagnetic term

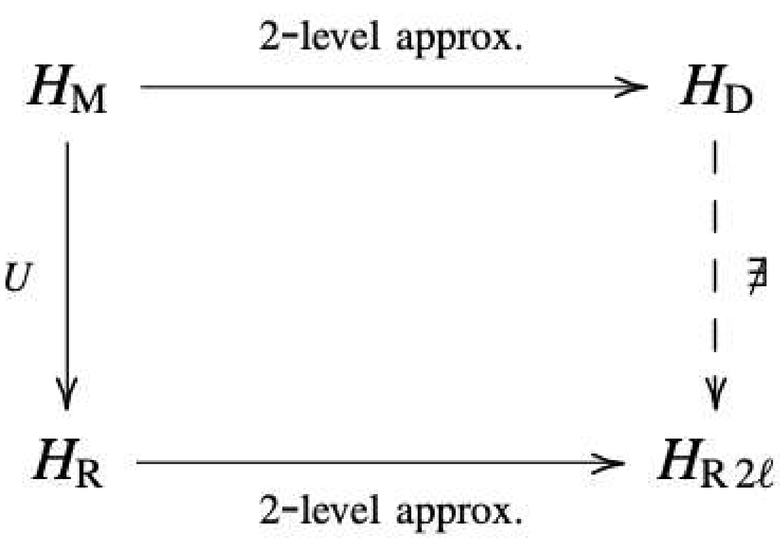
$$|\langle e|\vec{d}|g\rangle|^2 \omega_0 < \frac{e^2}{2m}$$

The coefficientes κ and γ are not independent. It would be impossible to have γ large enough to have a phase transition.

No-go theorem

What is the "correct" Hamiltonian?
If the diamagnetic term is present, there is NO ground state phase transtion

Mathematical methods in quantum optics: the Dicke model
E Nahmad-Achar, O Castaños, R López-Peña and J G Hirsch
Phys. Scr. **87** (2013) 038114
Fenómenos críticos en sistemas átomo-campo, Tesis doctoral de Miguel A. Bastarrachea Magnani, UNAM (2016).



The gauge invariance is broken

Phase Transitions, Two-Level Atoms, and the A^2 Term
K. Rzażewski, K. Wódkiewicz, and W. Żakowicz
Phys. Rev. Lett. **35**, 432 (1975)

Are super-radiant phase transitions possible?
J. M. Knight, Y. Aharonov, and G. T. C. Hsieh
Phys. Rev. A **17**, 1454 (1978)

No-go theorem concerning the superradiant phase transition in atomic systems
Iwo Bialynicki-Birula and Kazimierz Rzążewski
Phys. Rev. A **19**, 301 (1979)

No-go theorem for the superradiant phase transition without dipole approximation
Krzysztof Gawędzki and Kazimierz Rzążewski
Phys. Rev. A **23**, 2134 – Published 1 May 1981

Adequacy of the Dicke model in cavity QED: A counter-no-go statement
András Vukics and Peter Domokos
Phys. Rev. A **86**, 053807 – Published 6 November 2012

Origin and implications of an A2-like contribution in the quantization of circuit-QED systems
Moein Malekakhlagh and Hakan E. Türeci
Phys. Rev. A **93** 012120 (2016)

Relevance of the Quadratic Diamagnetic and Self-Polarization Terms in Cavity Quantum Electrodynamics
Christian Schäfer, Michael Ruggenthaler, Vasil Rokaj, and Angel Rubio
ACS Photonics **7** 975 (2020)

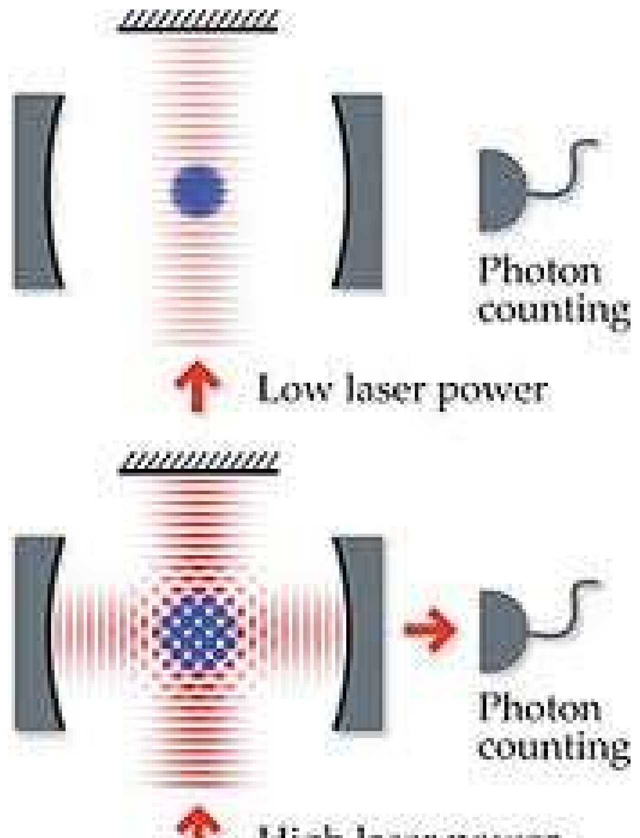
Dicke quantum phase transition with a superfluid gas in an optical cavity

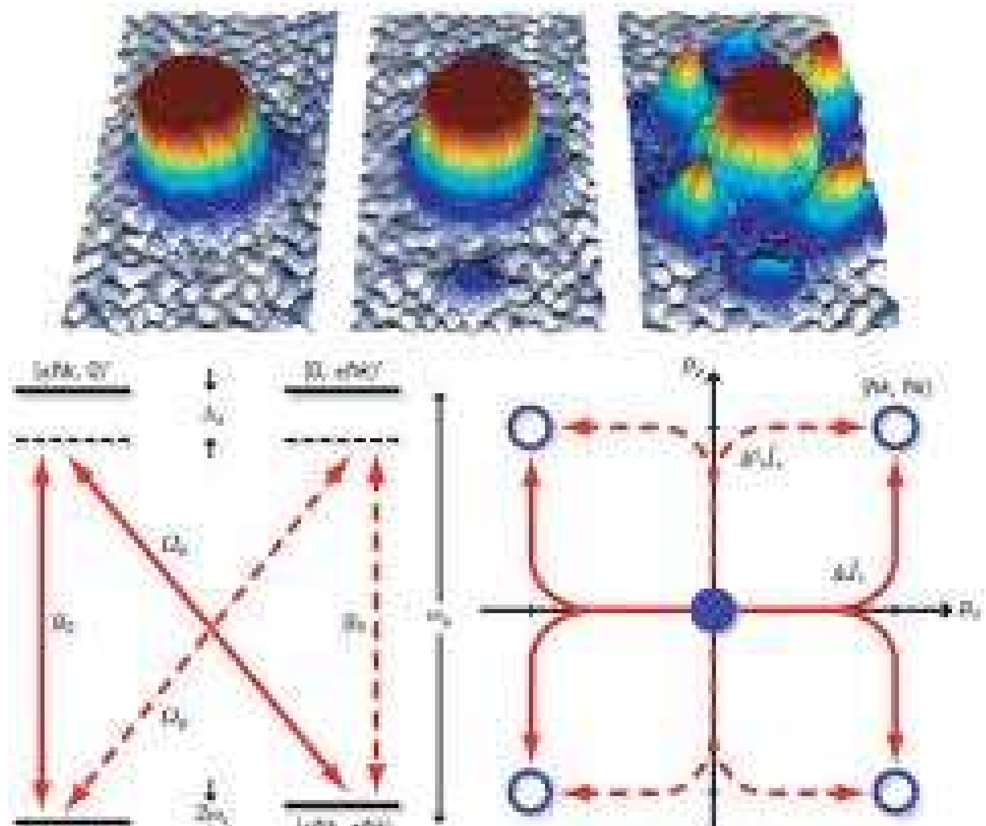
Kristian Baumann, Christine Guerlin, Ferdinand Brennecke & Tilman Esslinger

Nature **464**, 1301 (2010).

“We realize the Dicke quantum phase transition in an open system formed by a Bose–Einstein condensate coupled to an optical cavity, and observe the emergence of a self-organized supersolid phase.”

“We show that the phase transition is described by the Dicke Hamiltonian, including counter-rotating coupling terms, and that the supersolid phase is associated with a spontaneously broken spatial symmetry. The boundary of the phase transition is mapped out in quantitative agreement with the Dicke model.”





Rabi model (Only 1 atom)

The Jaynes–Cummings Model and Its Descendants. Modern research directions, Jonas Larson and Themistoklis Mavrogordatos, IOP Publishing Ltd 2021

$$\hat{H}_{\mathrm{R}} = \omega\hat{n} + \frac{\Omega}{2}\hat{\sigma}_z + g(\hat{a} + \hat{a}^\dagger)\hat{\sigma}_x. \qquad \omega = \Omega \text{ (in resonance)}$$

$$\frac{d\hat{\rho}}{dt} = \frac{1}{i\hbar}\left[\hat{H}_R, \hat{\rho}\right] + \kappa\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}a^\dagger\hat{a}\right)$$

Open system, with disipation

Table 1.1. Definition of the different regimes of the quantum Rabi model.

Regime	Parameters	Description
Weak coupling	$g/\omega \ll 1$ & $g < \kappa$	The coherent time-scale g^{-1} is the longest—the system evolution is dominated by losses.
Strong coupling	$g/\omega \ll 1$ & $g > \kappa$	The evolution for time-scales g^{-1} is predominantly unitary. The RWA is valid.
Ultrastrong coupling	$g/\omega \sim 0.1$ & $g > \kappa$	The RWA breaks down. Perturbation theory still captures the corrections. The TLA breaks down.
Deep strong coupling	$g/\omega \gtrsim 1$ & $g > \kappa$	The properties of the lower energy states of the model changes qualitatively, e.g. the expectation $\langle\hat{n}\rangle$ for the ground state is no longer approximately zero. The TLA breaks down.
Extreme strong coupling	$g/\omega \gtrsim 10$ & $g \gg \kappa$	Perturbative results hold in the polaron basis such that the light-matter degrees of freedom decouple. The TLA breaks down.

The Dicke Hamiltonian

$$H_D = \omega a^\dagger a + \omega_0 J_z + \frac{\gamma}{\sqrt{\mathcal{N}}}\left(a + a^\dagger\right)\left(J_+ + J_-\right).$$

Radiation mode: frequency ω, number operator $a^\dagger a$.
Atomic sector: excitation energy ω_0,
collective atomic pseudo-spin operators J_z, J_+, J_-
Number of atoms $\mathcal{N}$.
Symmetric atomic subspace, with eigenvalues of $\mathbf{J}^2$
restricted to $j(j+1)$, $j = \mathcal{N}/2$

$$\hbar = 1$$
$$\hbar_{eff} = 1/j$$

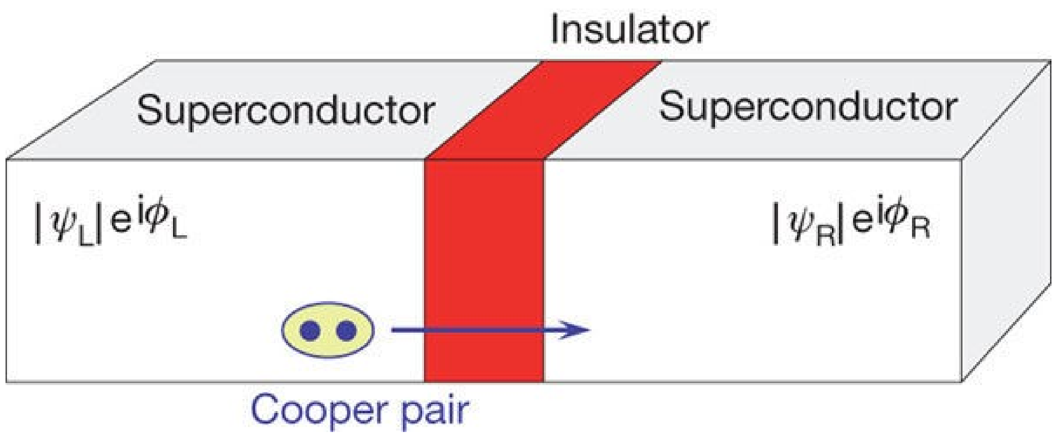
$$H_{TC} = \omega a^\dagger a + \omega_0 J_z + \frac{\gamma}{\sqrt{\mathcal{N}}}\left(aJ_+ + a^\dagger J_-\right).$$

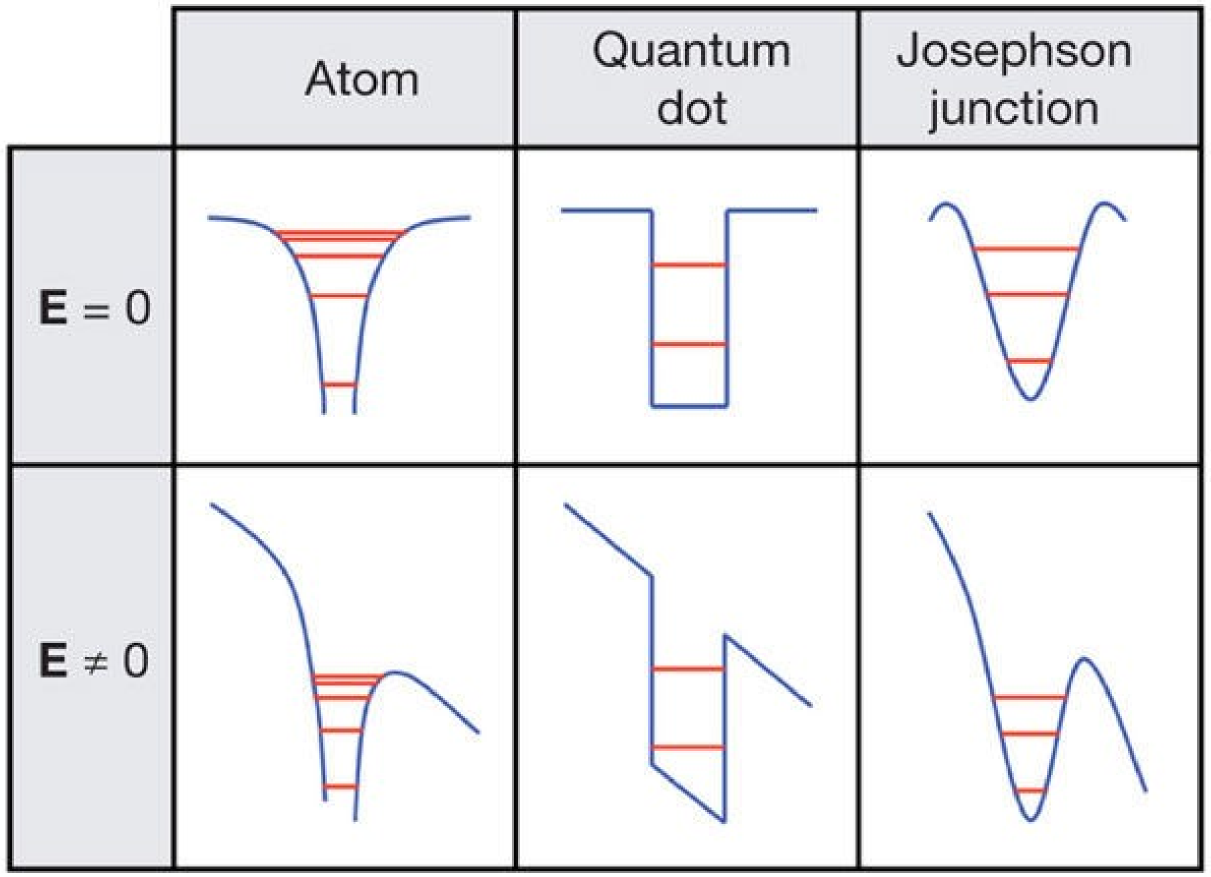
Tavis-Cummings: integrable version (RWA)

Superconducting circuits

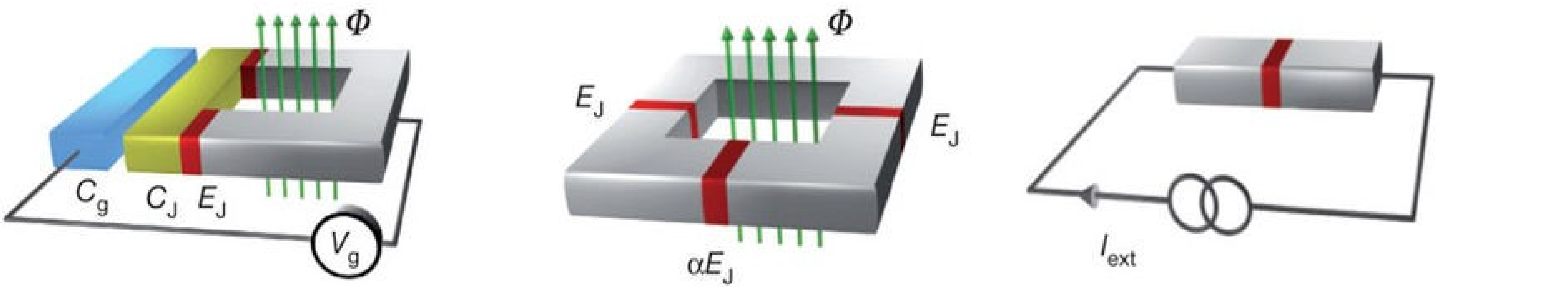
J. Q. You & Franco Nori, *Nature* **474**, 589–597 (2011)

Josephson junction





a Voltage-driven box (charge qubit) **b** Flux-driven loop (flux qubit) **c** Current-driven junction (phase qubit)



Classical Hamiltonian

Photon and atomic coherent states

$$
\begin{aligned}
|\alpha\rangle &= e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle, \\
|z\rangle &= \frac{1}{\left(1+|z|^2\right)^j} e^{z J_+} |j, -j\rangle
\end{aligned}
$$

photon vacuum $|0\rangle$

all atoms in their ground state $|j, -j\rangle$

Classical Hamiltonian

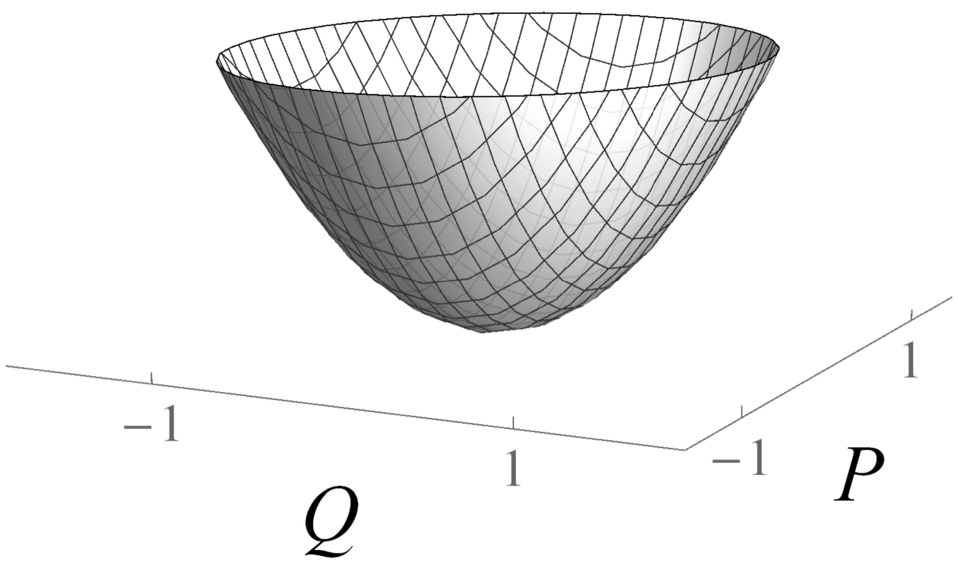
$$H_{cl} = \langle \alpha, z | H_D | \alpha, z \rangle / j$$
$$= \frac{\omega}{2}(p^2 + q^2) + \frac{\omega_0}{2}(P^2 + Q^2) + \gamma q Q \sqrt{4 - P^2 - Q^2} - \omega_0.$$

$$\alpha = \sqrt{\frac{j}{2}}(q + ip) \qquad z = \frac{Q - iP}{\sqrt{4 - (Q^2 + P^2)}}$$

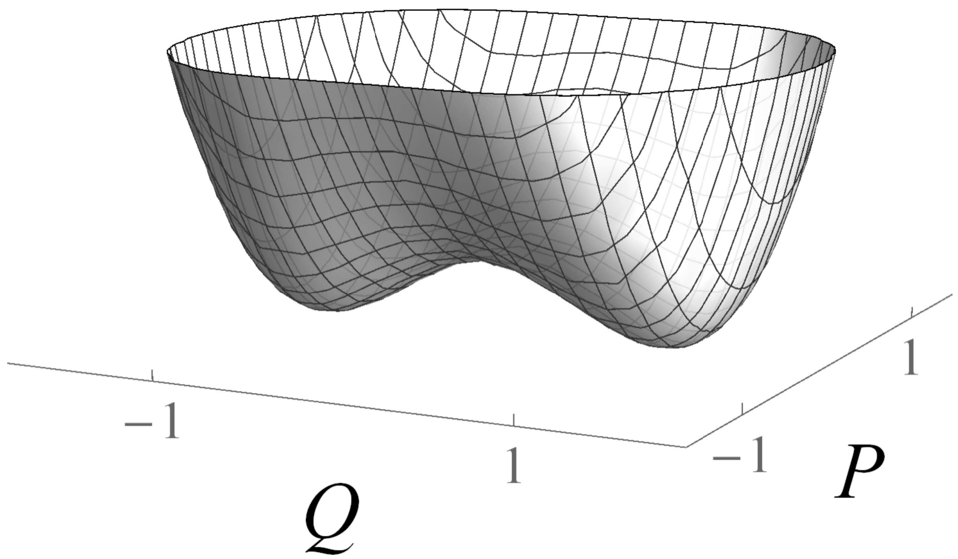
- C. Emary, T. Brandes, Phys. Rev. E **67**, 066203 (2003)
- O. Castaños, E. Nahmad-Achar, R. López-Peña, and J. G. Hirsch, Phys. Rev. A **83** 051601 (2011).
- O. Castaños, E. Nahmad-Achar, R. López-Peña, and J. G. Hirsch, Phys. Rev. A **84** 049901 (2011).
- L. Bakemeier, A. Alvermann, and H. Fehske, Phys. Rev. Lett. **88**, 043835 (2013).
- A. Altland, F. Haake, New J Phys. **14** 073011 (2014).

Energy surfaces in the classical model

$$E = H_{cl}(q, p, Q, P)$$



Normal phase

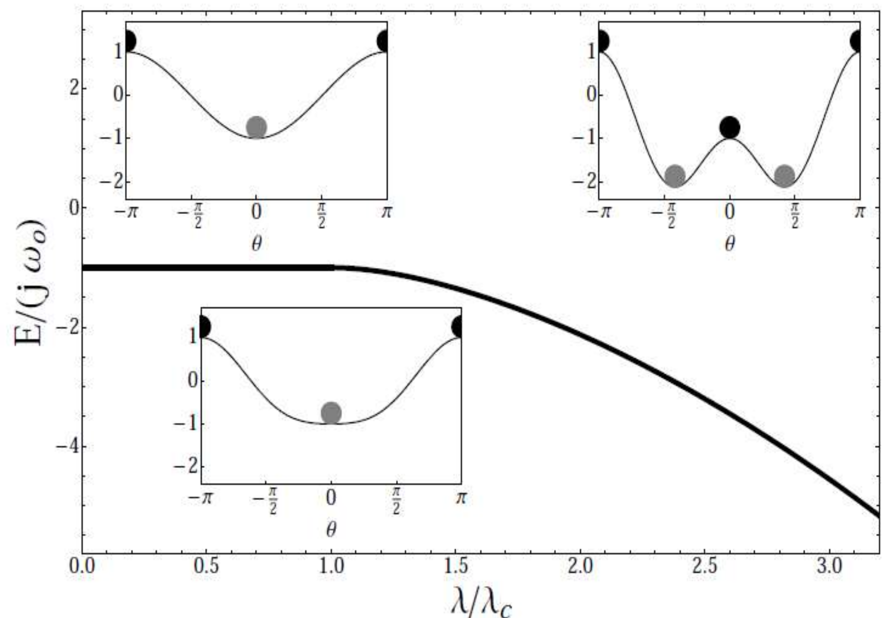


Superradiant phase

Quantum phase transition

- Zero temperature
- Second order P.T. as a function of a Hamiltonian parameter
- An abrupt change in the ground state of a many-body system due to its quantum fluctuations
- Order parameters exhibit discontinuities in derivatives in the thermodynamic limit.

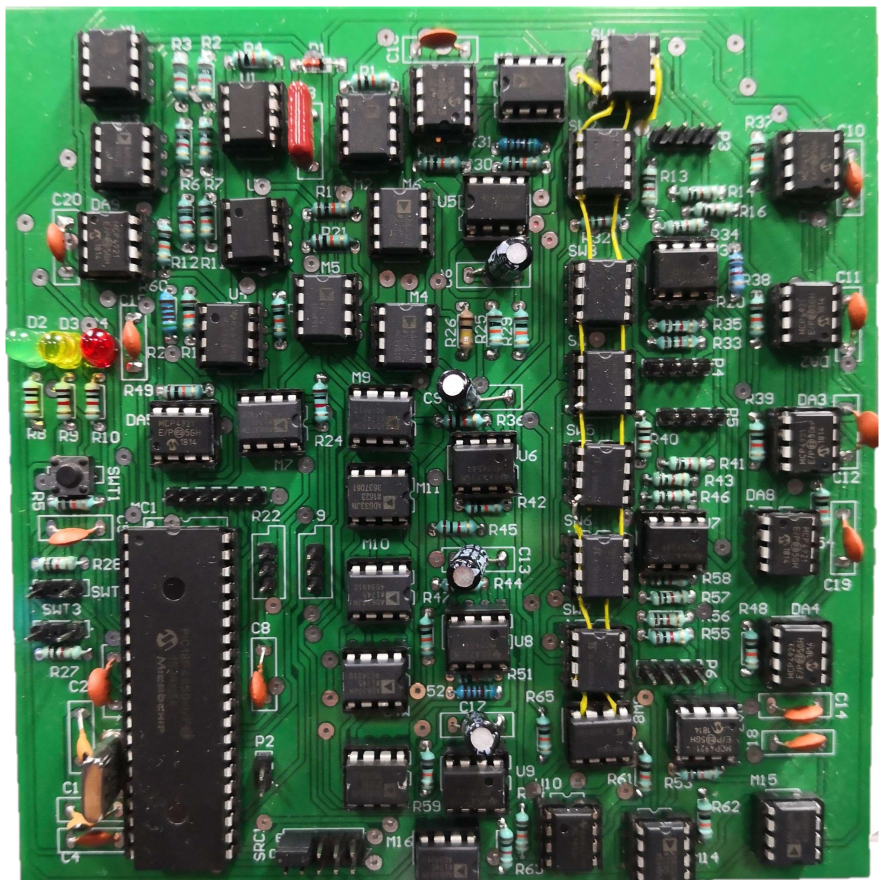
- *Sachdev, Subir (2011). Quantum Phase Transitions. Cambridge University Press. (2nd ed.).*

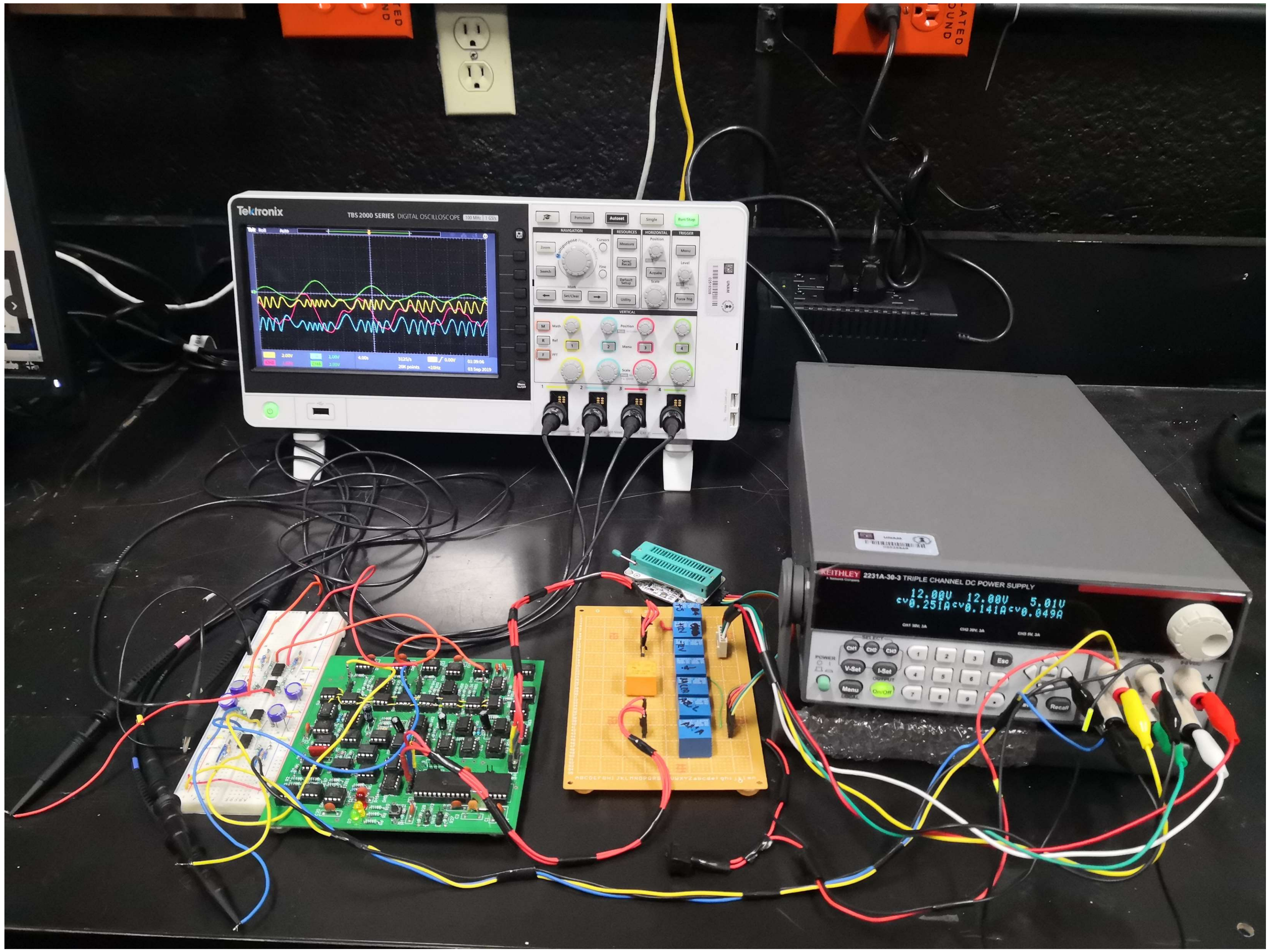


It has been electronically implemented at ICN with a nonlinear network of electrical oscillators with analog electrical components

with Mario Quiroz, Jorge Chávez and Roberto de León, Phys. Rev. Res. 2 (2020) 033169

The voltages in the oscilloscope are proportional to (q,p,Q,P)

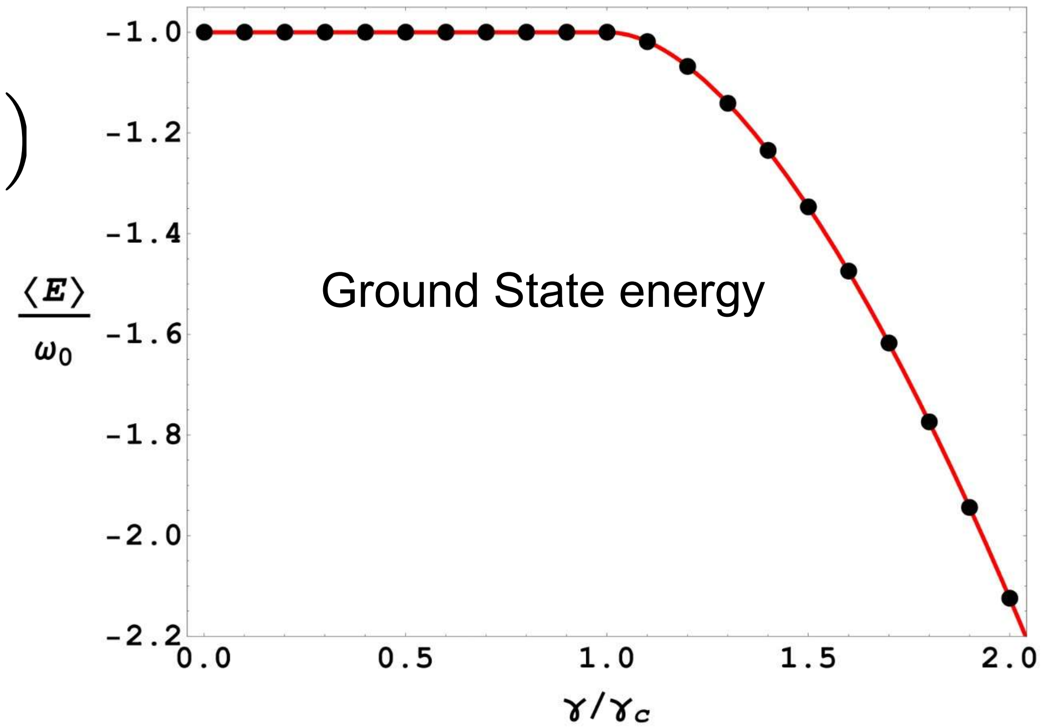




A mean field Quantum Phase Transition

with Mario Quiroz, Jorge Chávez and Roberto de León, Phys. Rev. Res. 2 (2020) 033169

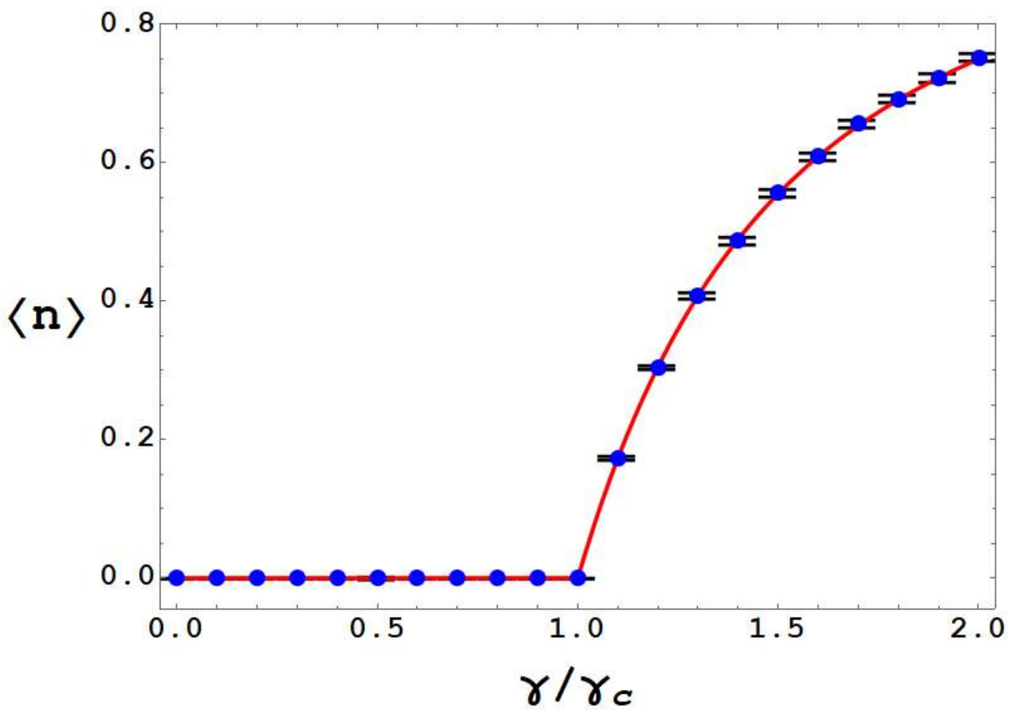
$$E = H_{cl}(q, p, Q, P)$$



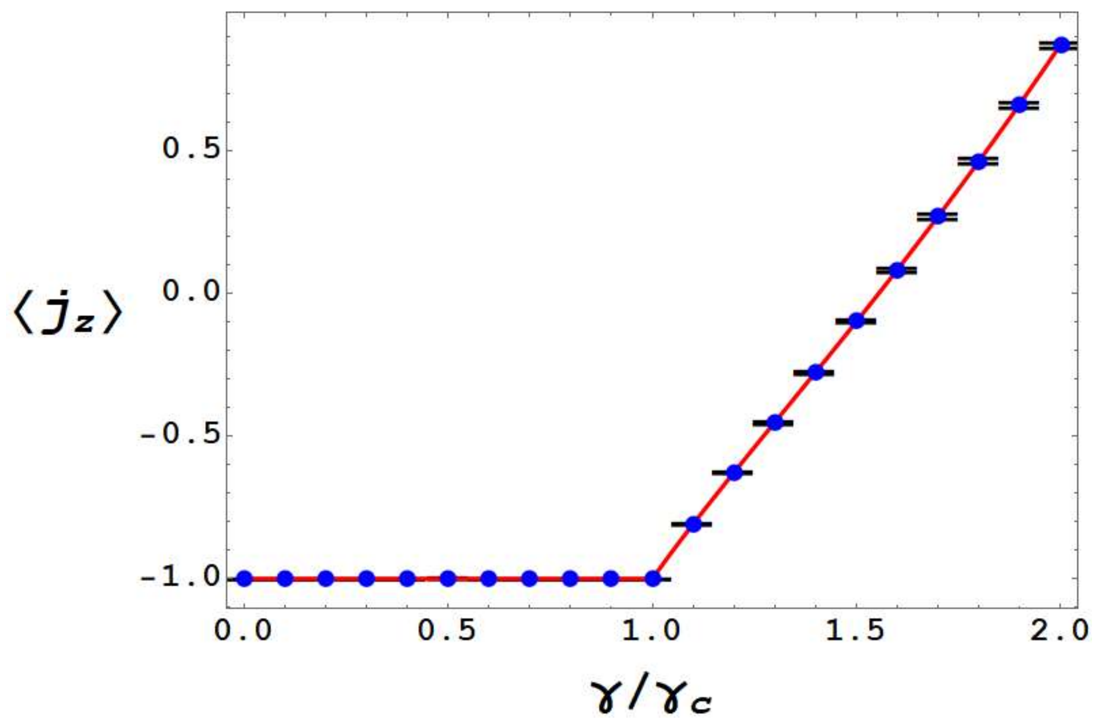
Electronic Realization

Experimental voltages in the oscilloscope compared with theory

$$\langle n \rangle = (q^2 + p^2)/2$$



$$\langle j_z \rangle = (Q^2 + P^2)/2$$



The theoretical description of the observed ground state phase transition predicted a **divergence** in the expectation value of the number of photons and the number of excited atoms

F. Dimer, B. Estienne, A.S. Parkins, and H.J. Carmichael, Phys. Rev. **A 75**, 013804 (2007) .
D. Nagy, G. Konya, G. Szirmaind P. Domokos, Phys. Rev. Lett. **104**, 130401 (2010).

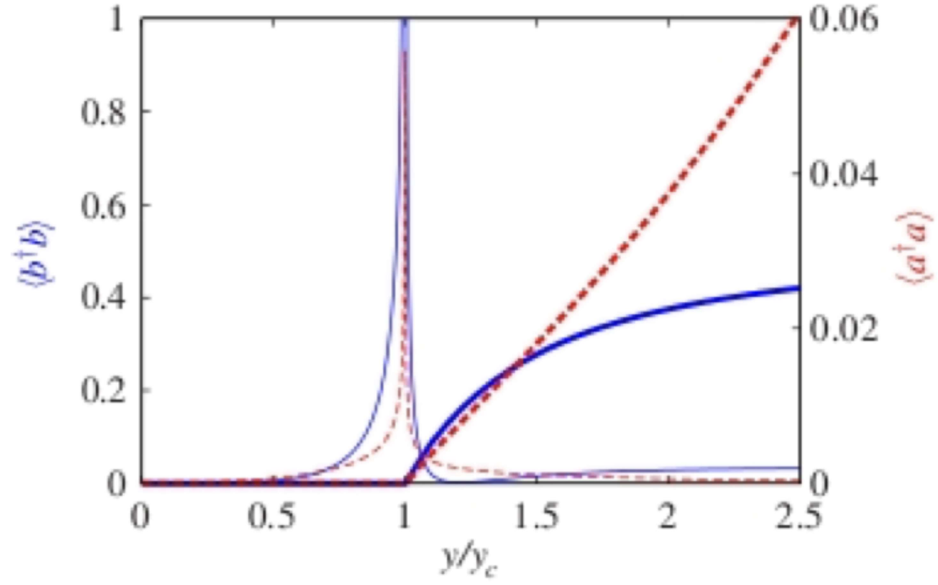


FIG. 1 (color online). Photon (dashed red lines) and motionally excited atom (solid blue lines) numbers in the ground state.

Where do the divergences come from?

- From a mixing of intensive and extensive observables.
- From a truncation in the series expansion of the angular momentum operators, which provides truncated versions of the Hamiltonian.

It is assumed that $\frac{b^\dagger b}{2j} \approx \frac{<b^\dagger b>}{2j} \approx 0$

Holstein Primakoff realization of the rotor algebra

$$J_+ = \sqrt{2j}\, b^\dagger \sqrt{1 - \frac{b^\dagger b}{2j}}, \qquad J_- = \sqrt{2j}\sqrt{1 - \frac{b^\dagger b}{2j}}\, b$$

$$J_z = b^\dagger b - j \qquad \text{with} \quad j = N/2$$

The truncated Hamiltonian is quadratic in the Bosonic fields

$$H_{trunc} = \omega a^\dagger a + \omega_0 (b^\dagger b - j) + \gamma (a^\dagger + a)(b^\dagger + b)$$

No singularities in observables at the phase transition in the Dicke model,
O. Castaños, E. Nahmad-Achar, R. López-Peña, and J. G. Hirsch, Phys. Rev. A 83 (2011) 051601(R).

Virtues and limitations of the truncated Holstein-Primakoff description of quantum rotors,

Regular orbits $\sigma^2 \approx \hbar_{eff} = 1/j$

(TWA with classical trajectories, Gaussian ensemble)

D. Villaseñor, et al, New Journal of Physics **22** (2020) 063036

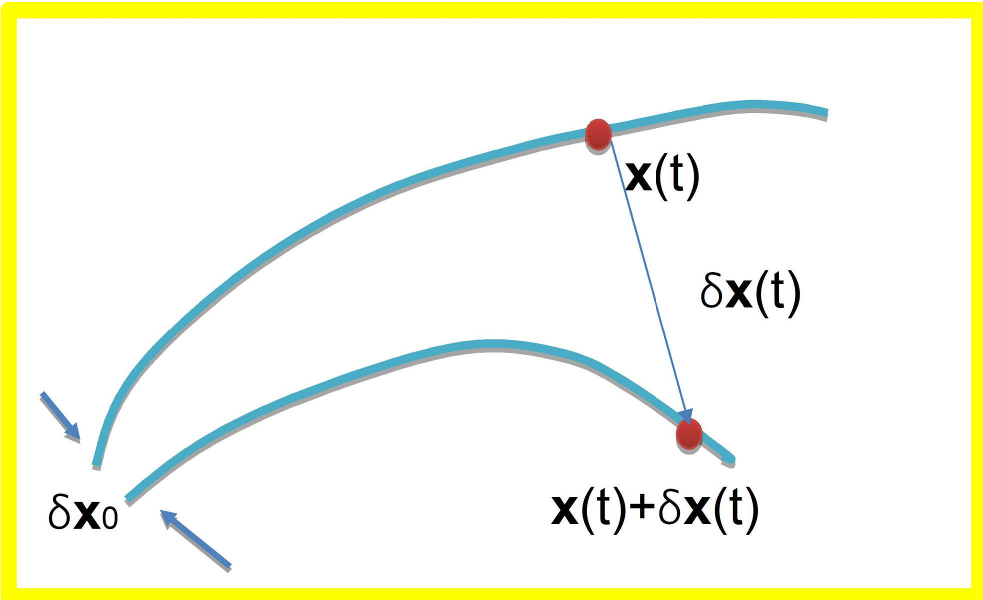
Chaotic orbits $\sigma^2 \approx \hbar_{eff} = 1/j$

(TWA with classical trajectories, Gaussian ensemble)

D. Villaseñor, et al, New Journal of Physics **22** (2020) 063036

LYAPUNOV EXPONENT

Given a perturbation $\delta\mathbf{x}_0$ of $\mathbf{x}_0$, the new path is $\mathbf{x(t)}$



Change in the path: $|\delta\mathbf{x}(t)| \approx e^{\lambda t}\,|\delta\mathbf{x}_0|$

$$\lambda = \lim_{t\to\infty}\ \lim_{\|\delta\mathbf{x}_0\|\to 0} \frac{1}{t} \ln \frac{\|\delta\mathbf{x}(t)\|}{\|\delta\mathbf{x}_0\|}$$

The parameter λ is the <u>*maximum Lyapunov exponent*</u>

$\lambda = 0$ "Regular"

$\lambda < 0$ "Regular" & "Atractor"

$\lambda > 0$ "Chaos"

QUANTITATIVE MEASURE OF CHAOS.

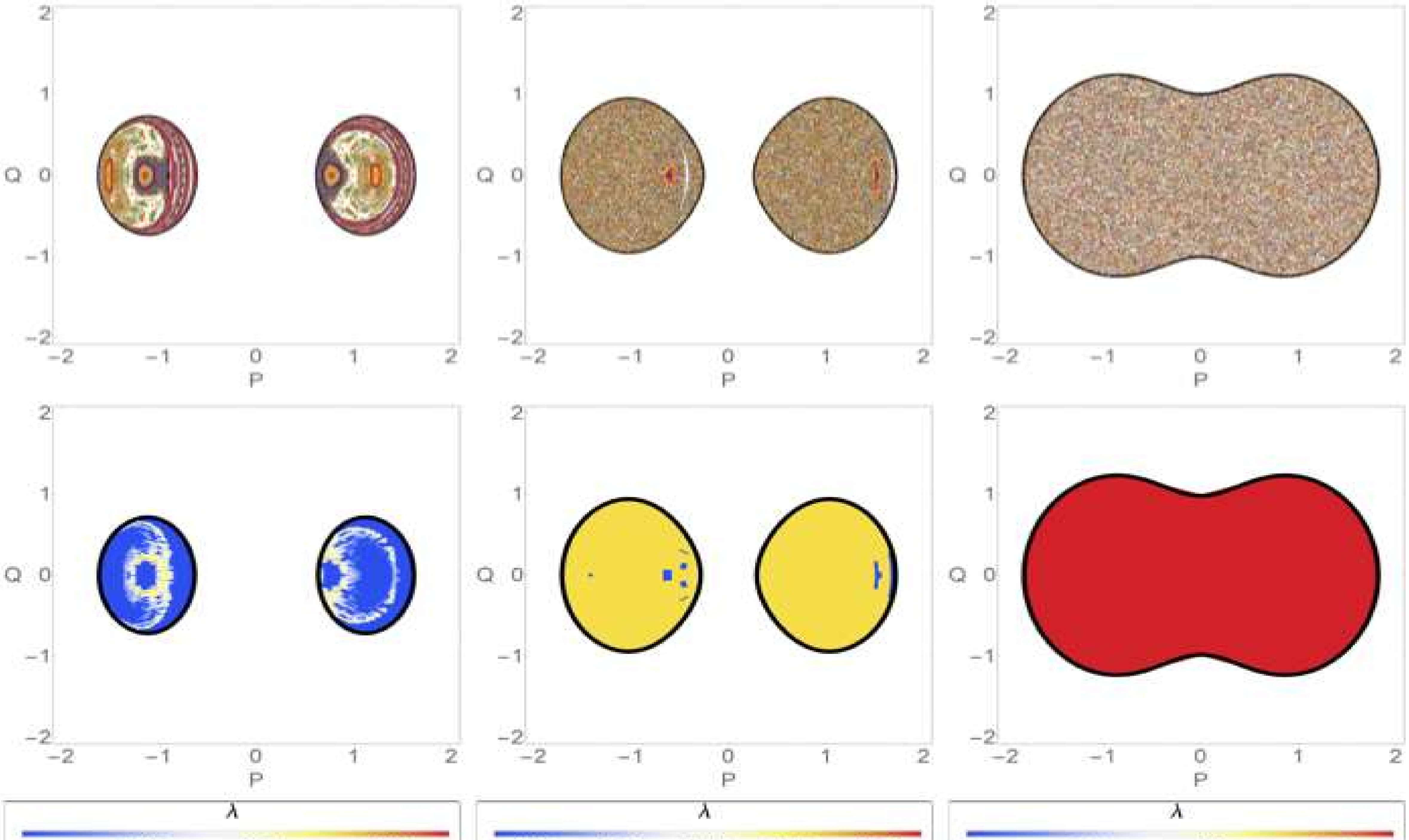
$t_\lambda = 1/\lambda$ Lyapunov time

- M. A. Lyapunov, The general problem of the stability of motion (in Russian). Kharkov Mathematical Society 250 pp (1892).
- Ch. Skokos, The Lyapunov Characteristic Exponents and Their Computation. Lect. Notes Phys. 790, 63-135 (2010).

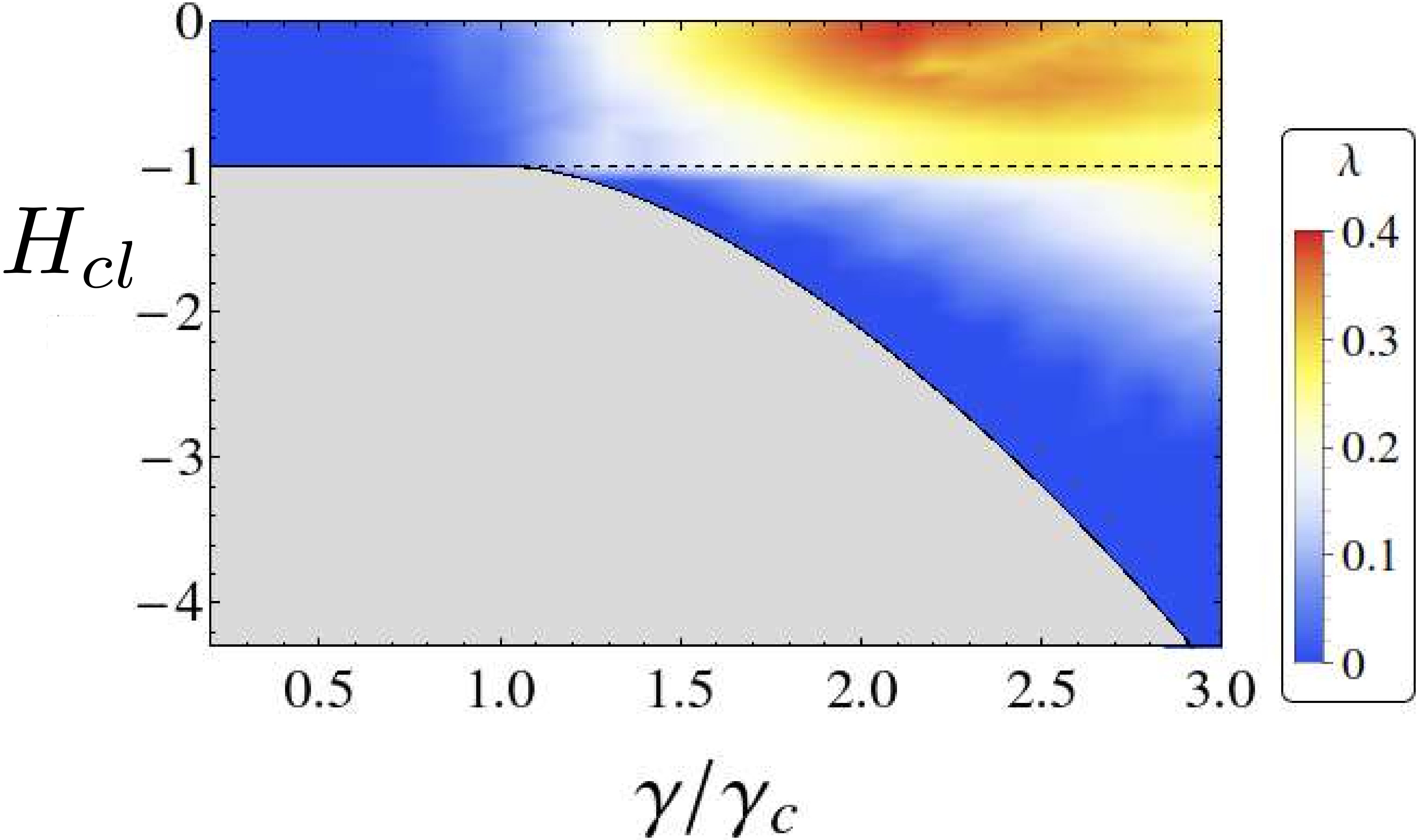
Poincaré surface sections and Lyapunov exponents

$$\omega = \omega_0, \gamma = 2\gamma_c$$

$\epsilon = -1.4\omega_0$ $\epsilon = -1.1\omega_0$ $\epsilon = -0.5\omega_0$



Average Lyapunov exponent



- Jorge Chávez-Carlos, Miguel Angel Bastarrachea-Magnani, Sergio Lerma-Hernández and Jorge G Hirsch, Phys. Rev. E **94**, (2016) 022209.

Quantum description

The Hilbert space is infinite $\longrightarrow$
Truncation in the maximum number of photons

Efficient basis: rotation $J_z = -J'_x$, $J_x = J'_z$ with $J_x = \frac{J_+ + J_-}{2}$.

displaced photons $A = a + \frac{2\gamma}{\omega\sqrt{N}} J'_z = a + G J'_z$.

$$H_D = \omega \left(A^\dagger A - G^2 {J'}_z^2 \right) - \frac{\omega_0}{2} \left(J'_+ + J'_- \right).$$

With this coherent basis more than 50,000 converged states have been calculated for systems with 200 atoms.

M. A. Bastarrachea-Magnani, et. al., Rev. Mex. Fis. S **57**, 0069 (2011).
M. A. Bastarrachea-Magnani, et. al., AIP Conf. Proc. **1488**, 418 (2012).
M. A. Bastarrachea-Magnani, et. al., Phys. Scr. **T160,** 014005 (2014);
T160, 014018 (2014).

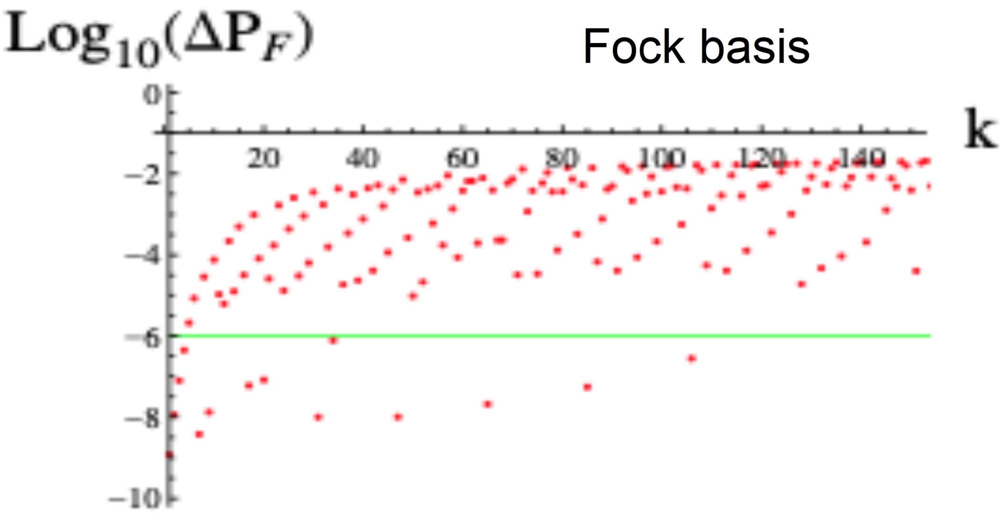
How efficient is the efficient basis?

Convergence measure

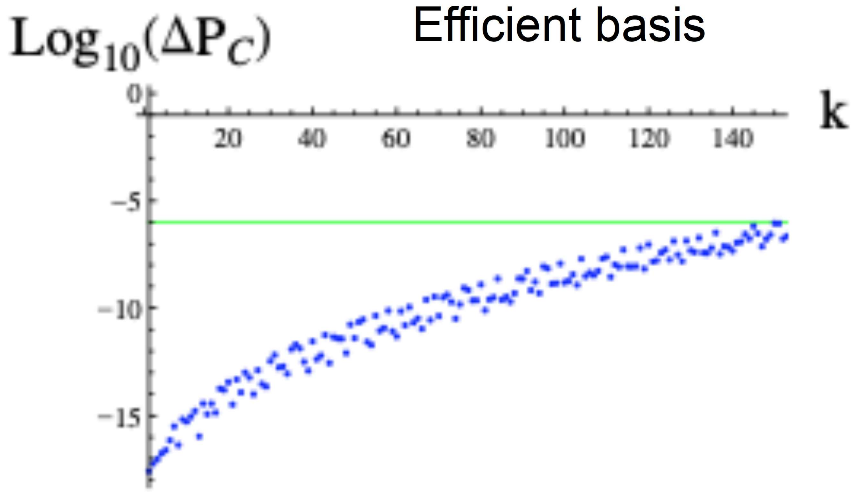
Efficient basis for the Dicke Model II: wave function convergence and excited states,
Jorge G. Hirsch and Miguel Angel Bastarrachea-Magnani Phys. Scr. T160 (2014) 014018

$$\Delta P_X \leqslant \sum_{m=-j}^{j} \left| C^{1,X}_{x_{\max}+1,m}(x_{\max}+1) \right|^2 .$$

Fock basis



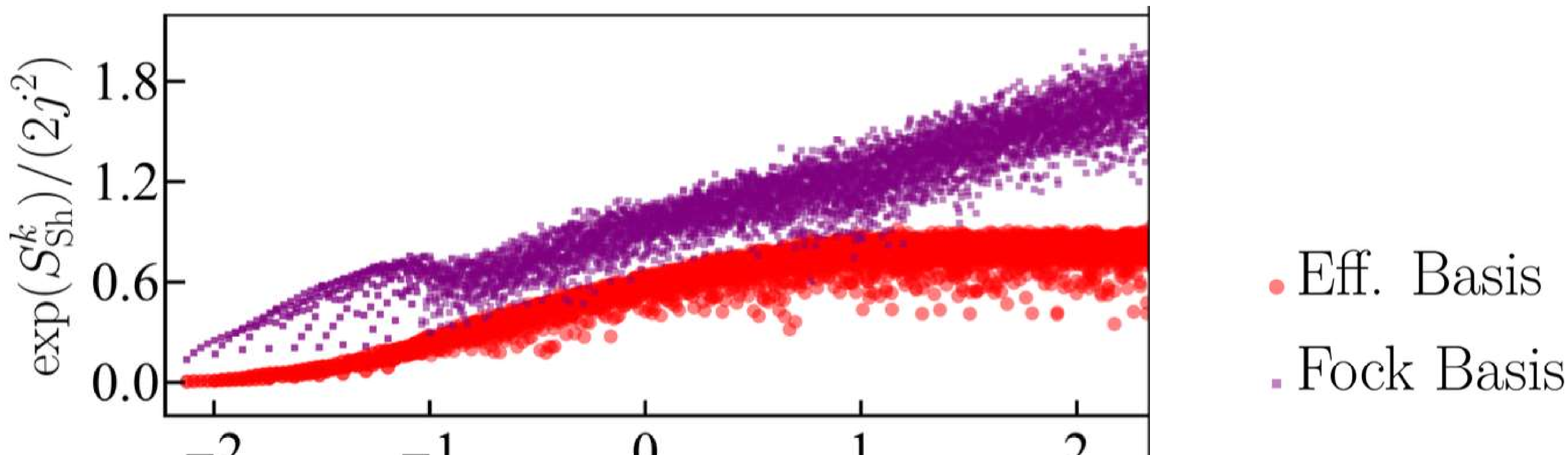
Efficient basis



$j = 40,\ N_{max} = 20$

Effective number of states in the basis, needed to describe converged eigenstates

Chaos and Thermalization in the Spin-Boson Dicke Model,
D. Villaseñor, S. Pilatowsky-Cameo, M. A. Bastarrachea-Magnani, S. Lerma- Hernández, Lea F. Santos and J. G. Hirsch, Entropy 25 (2023) 8.



Density of states

M.A. Bastarrachea-Magnani, S. Lerma-Hernández and J. G. Hirsch, Phys. Rev. A **89** (2014) 032101, Phys. Rev. A **89** (2014) 032102
M.A. Bastarrachea-Magnani, B. López-del-Carpio, S. Lerma-Hernández and J. G. Hirsch, Phys. Scr. 90 (2015) 068015

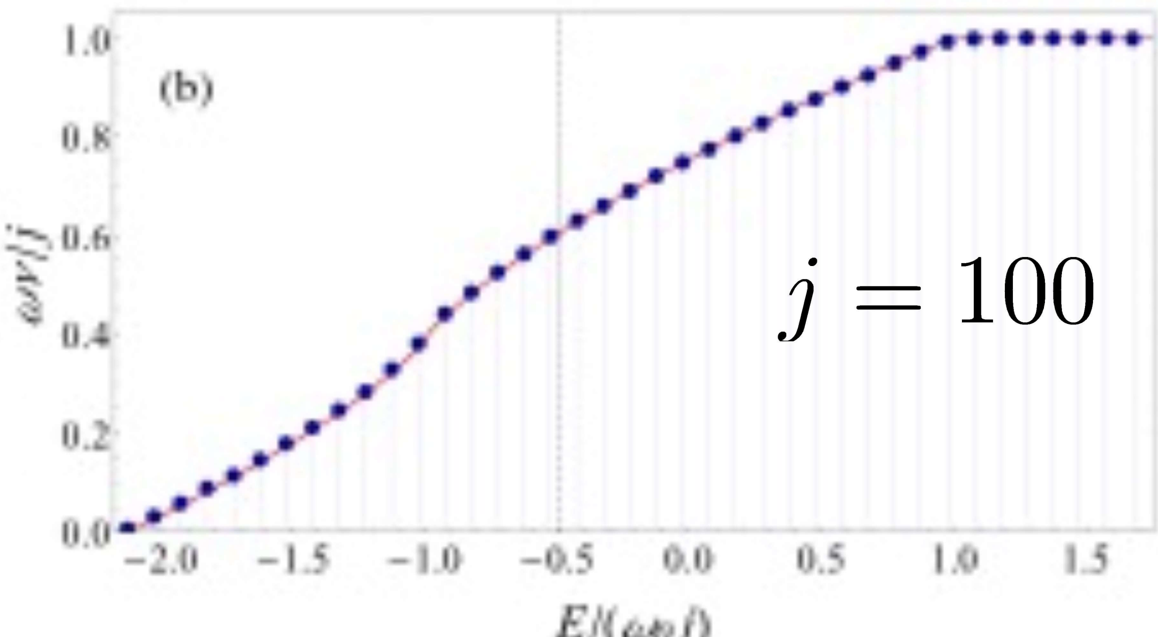
Analytical expression

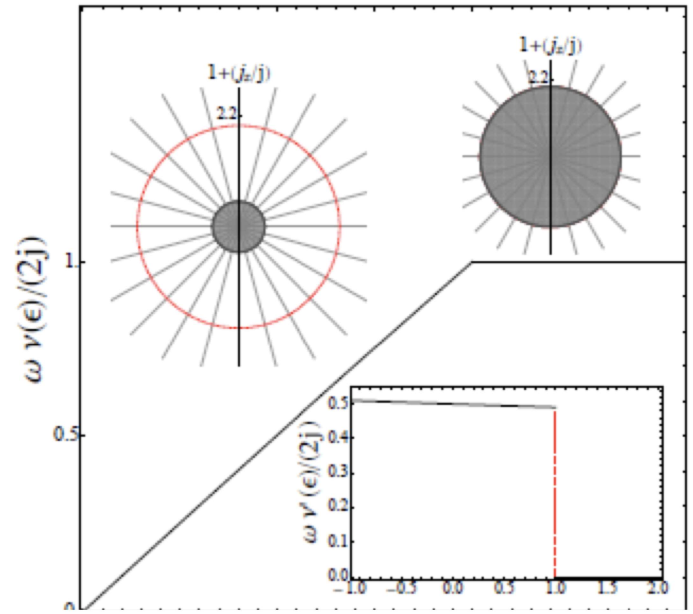
$$\frac{\omega}{2j}\nu(\epsilon) = \begin{cases} \frac{1}{\pi}\int_{y_-}^{y_+} \arccos\sqrt{\frac{2\gamma_c^2(y-\epsilon)}{\gamma^2(1-y^2)}}\,dy, & \epsilon_0 \leqslant \epsilon \leqslant -1, \\ \frac{\epsilon+1}{2} + \frac{1}{\pi}\int_{\epsilon}^{y_+} \arccos\sqrt{\frac{2\gamma_c^2(y-\epsilon)}{\gamma^2(1-y^2)}}\,dy, & |\epsilon| < 1, \\ 1, & \epsilon \geqslant 1, \end{cases}$$

$$y_\pm = \left(-\frac{\gamma_c^2}{\gamma^2} \pm \frac{\gamma_c}{\gamma}\sqrt{2(\epsilon-\epsilon_0)}\right) \text{ and } \epsilon \equiv \frac{E}{\omega_o j},\ \epsilon_0 \equiv \frac{E_{min}}{\omega_o j}$$

$$\omega = \omega_0, \gamma = 2\gamma_c$$

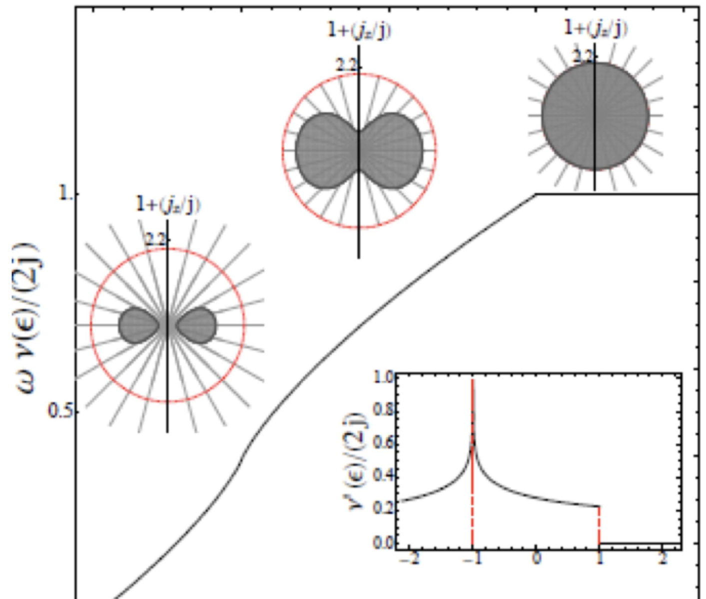
Excellent agreement with the exact energy spectra.
Allows unfolding without any parameter.





$\gamma \leq \gamma_c$

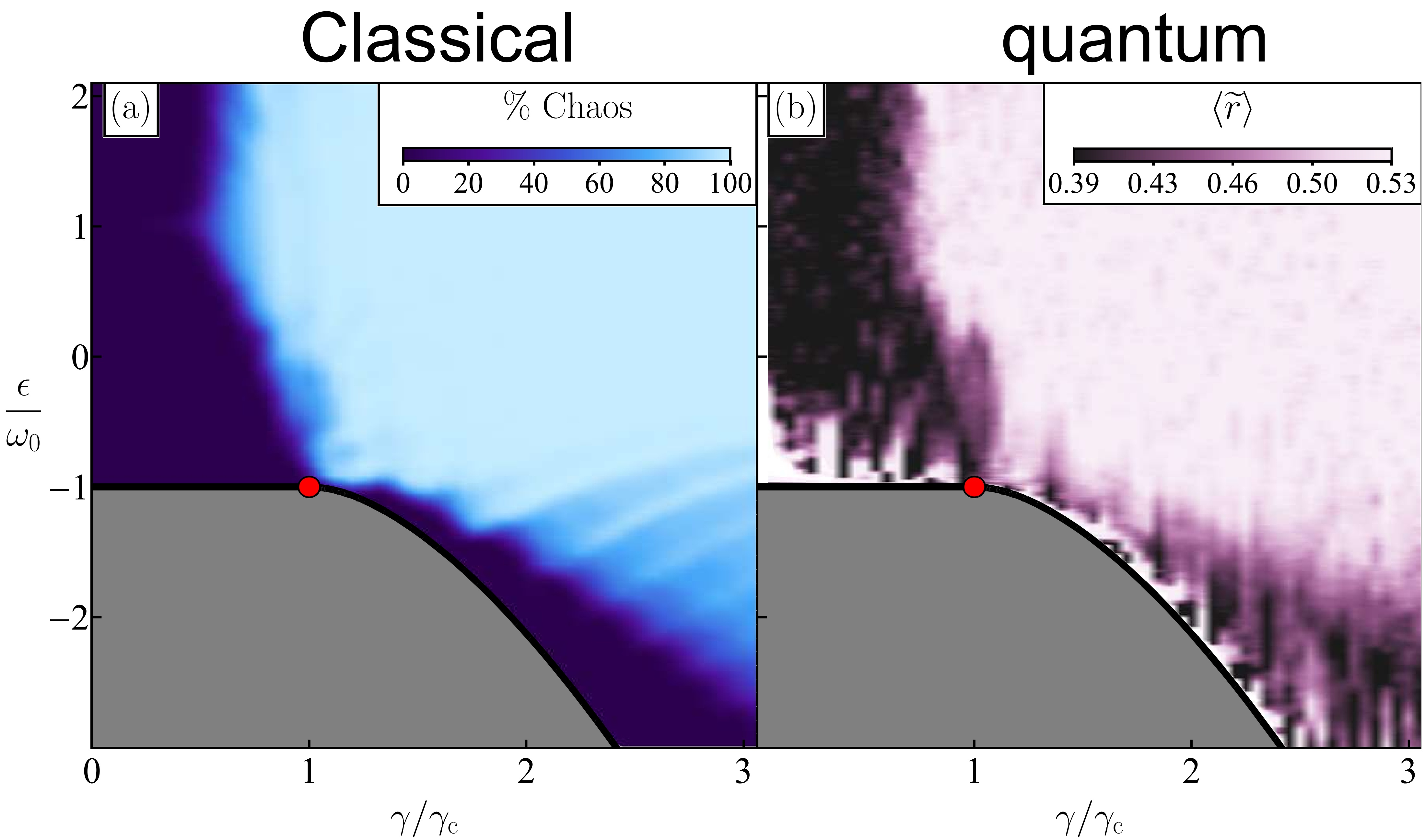
ESQPT



$\gamma > \gamma_c$

Fraction of classical chaotic orbits vs. quantum <r>

Chaos and Thermalization in the Spin-Boson Dicke Model,
D. Villaseñor, S. Pilatowsky-Cameo, M. A. Bastarrachea-Magnani, S. Lerma- Hernández, Lea F. Santo and J. G. Hirsch, Entropy 25 (2023) 8.



Participation Ratio

of a coherent state in the eigenstate basis.

$$P_R = \frac{1}{\sum_{k=1}^{N} |\langle \phi_k | \Psi \rangle|^4}.$$

Localized $P_R = 1$

De-localized $P_R = N$

A point in phase space

$$(q^0, p^0, j_z^0, \phi^0)$$

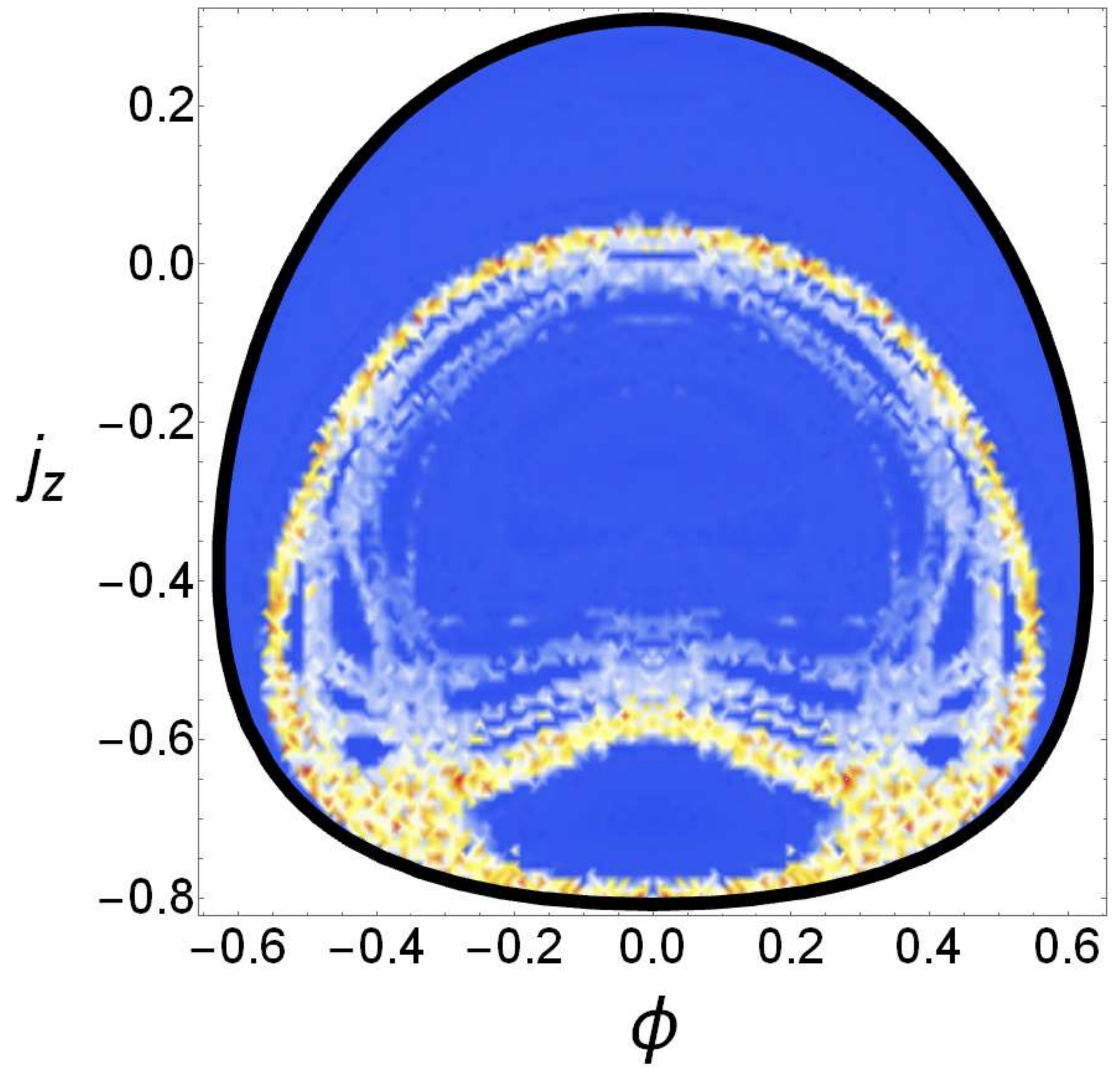
At a given energy

$$q^0 = q_{\pm}(\epsilon, p^0, j_z^0, \phi^0)$$

$$P_R = \frac{1}{\sum_k |\langle E_k | \alpha_0, z_0 \rangle|^4} = \frac{1}{\sum_k Q_k^2(\alpha_0, z_0)}.$$

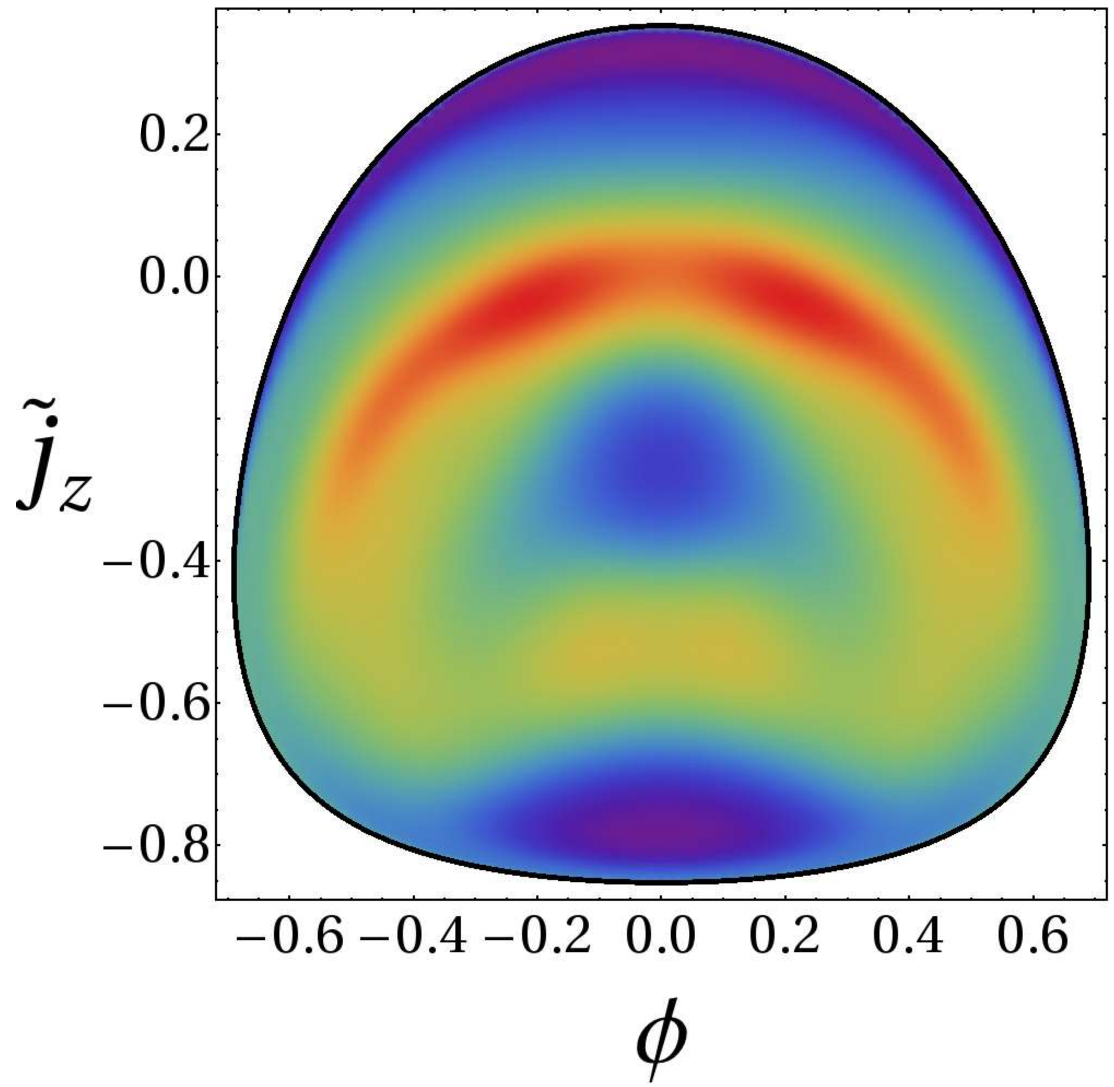
Lyapunov exponent vs Participation Ratio of a coherent state in the eigenbasis

Classical



Lyapunov exponent **(λ)** in phase space,
$\gamma = 2\gamma_c$, $\epsilon = -1.5\omega_0$, $\omega=\omega_0$.

Quantum



Participation Ratio **(P_R)** of coherent states in phase space,
$\gamma = 2\gamma_c$, j=100, $\epsilon = -1.5\omega_0$, $\omega=\omega_0$.

M.A. Bastarrachea-Magnani, B. López-del-Carpio, J. Chávez-Carlos, S. Lerma-Hernández and J. G. Hirsch, Phys. Rev. E 93 (2016) 022215.
M. A. Bastarrachea-Magnani, J. Chávez-Carlos, S. Lerma-Hernández, J. G. Hirsch,

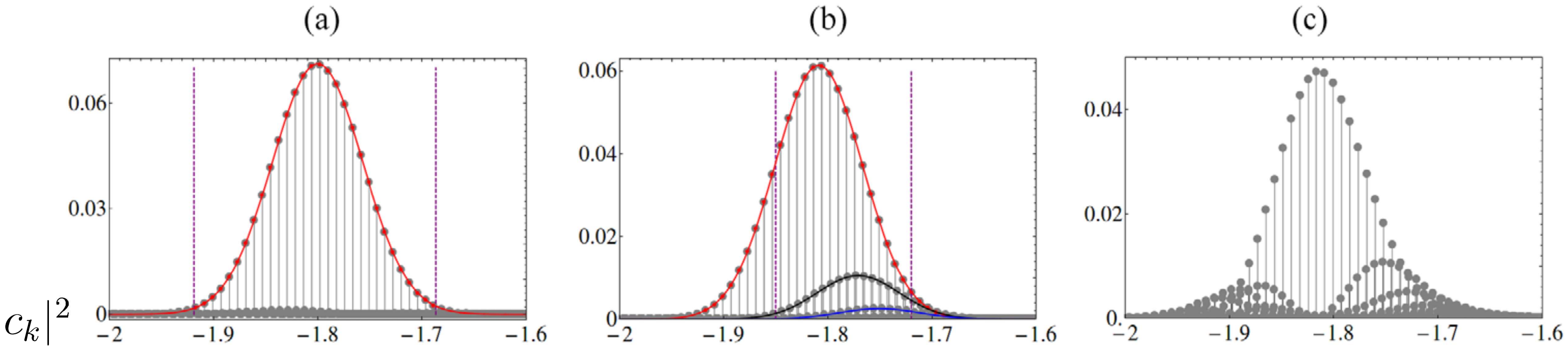
Survival probability

$$S_P(t) = |\langle\Psi(0)|\Psi(t)\rangle|^2 = S_P(t) = \left|\sum_k |c_k|^2 e^{-iE_k t}\right|^2$$

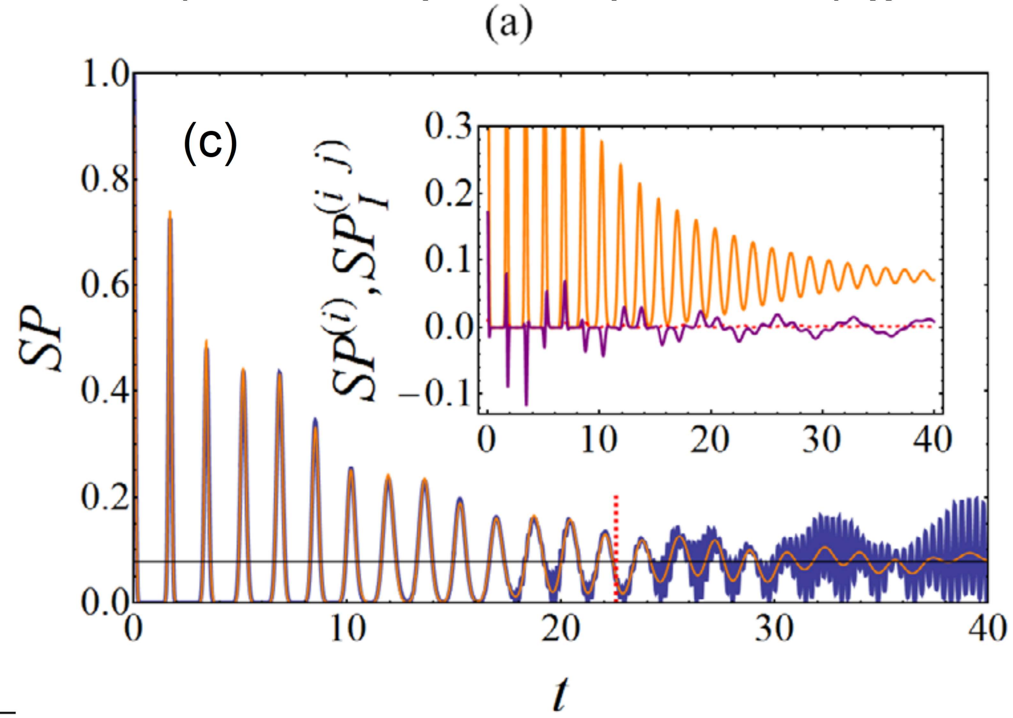
$$SP(t) \approx \frac{\omega_1}{2\sigma\sqrt{\pi}}\left\{1 + 2\sum_{p=1} \exp\left[-p^2\left(\frac{\omega_1^2}{4\sigma^2} + \frac{t^2}{t_D^2}\right)\right]\cos(p\omega_1 t)\right\}$$

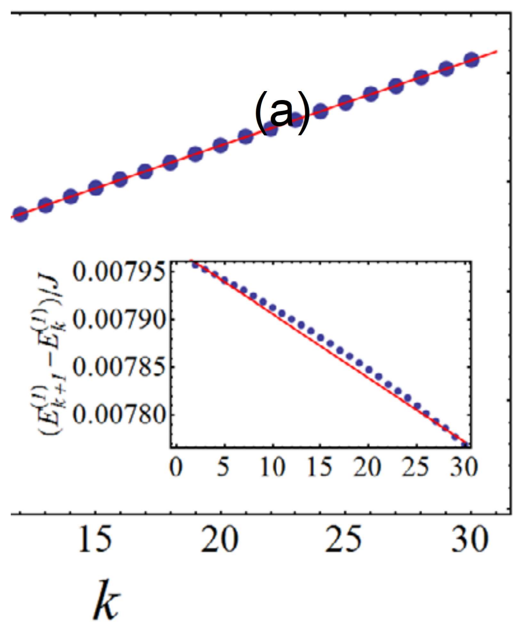
S. Lerma-Hernández, J. Chávez-Carlos, M.A. Bastarrachea-Magnani, L.F. Santos and J.G. Hirsch
J. Phys. A: Math. Theor. **51** (2018) 475302

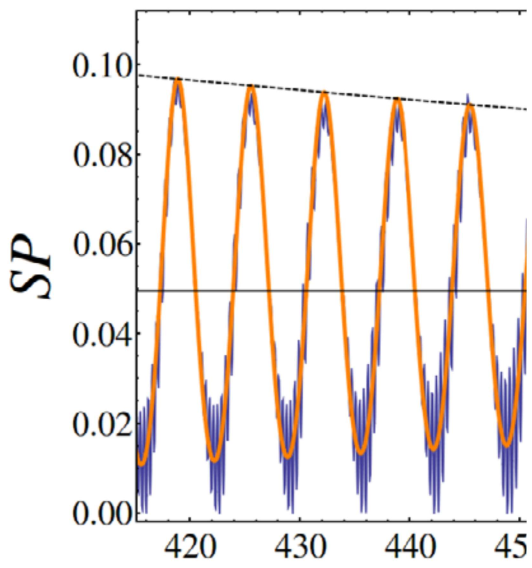
Eigenenergy components of a sample of coherent states with the same mean energy ($E/J = -1.8$) in the regular region

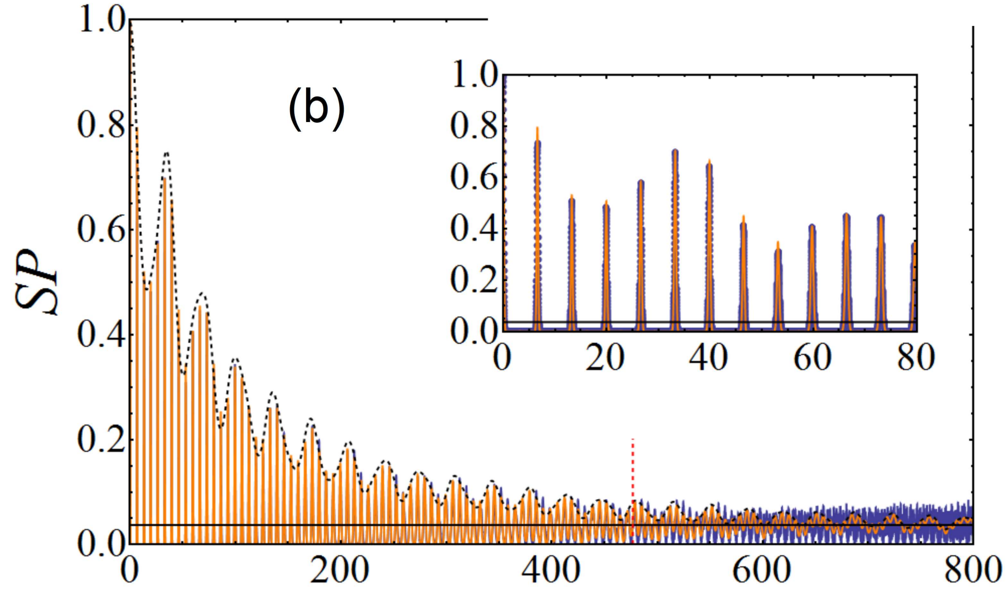


Survival probability for the coherent state numerical curve (dark blue) and analytical expression (light orange).









Survival probability in the chaotic region: correlation hole

Y. Alhassid and R. D. Levine, Phys. Rev. A 46, 4650–4653 (1992).
E.J. Torres-Herrera, A.M. García-García, and L.F. Santos, Phys. Rev. B 97, 060303(R) (2018).

The components of the initial state are selected as random numbers.

The level statistics is comparable to that of random matrices from Gaussian orthogonal ensembles (GOE)

$$\langle S_P(t)\rangle = \frac{1-\langle I_{PR}\rangle}{\eta - 1}\left[\eta S_P^{bc}(t) - b_2\left(\frac{t}{2\pi\nu_c}\right)\right] + \langle I_{PR}\rangle$$

$$\eta \equiv \frac{\nu_c}{\int \rho^2(E)dE} = \frac{\langle r_k^2\rangle}{\langle r_k\rangle^2}\frac{1}{\langle I_{PR}\rangle}$$

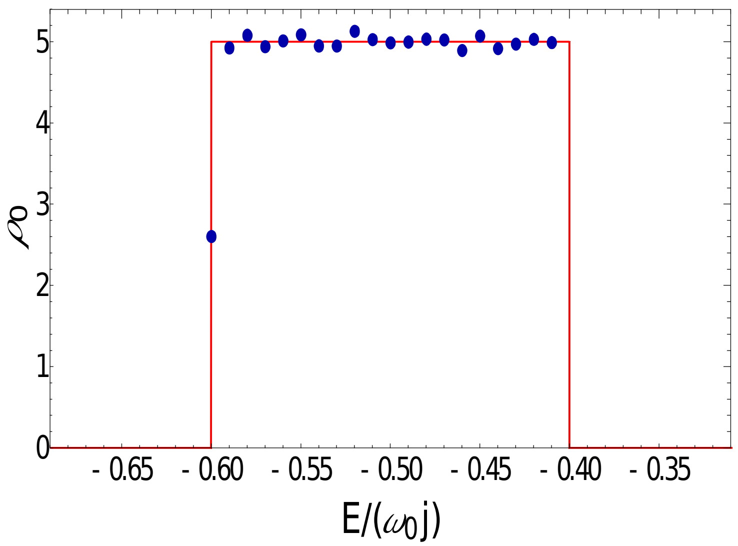
effective dimension of the ensemble

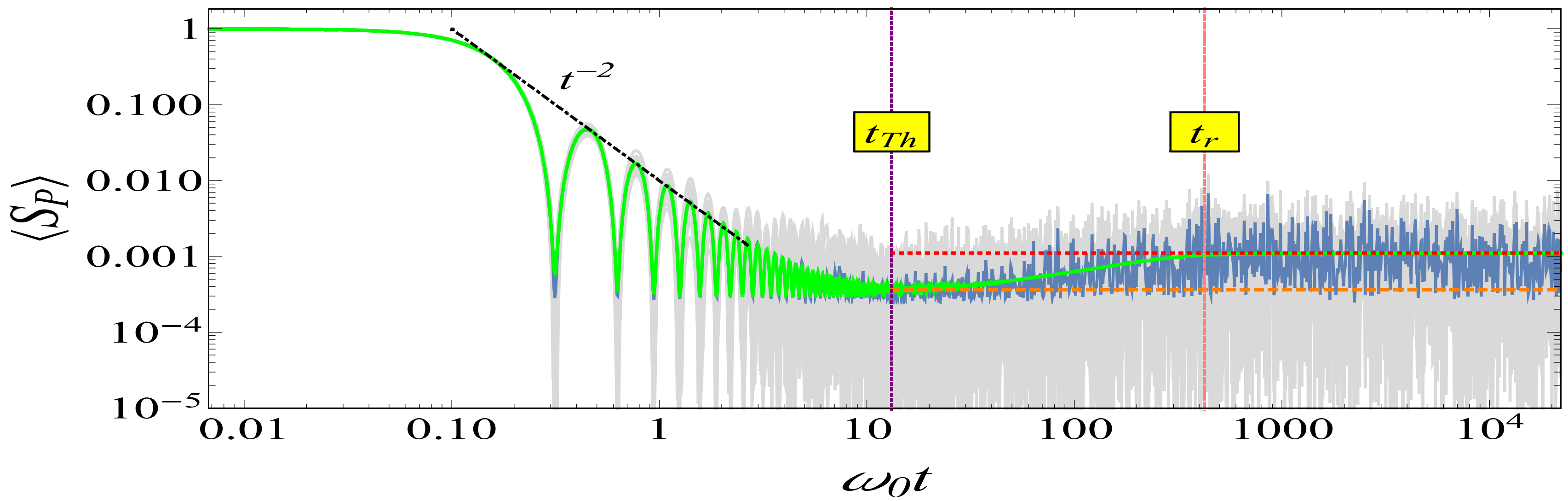
$$b_2(\bar{t}) = [1 - 2\bar{t} + \bar{t}\ln(2\bar{t}+1)]\Theta(1-\bar{t}) + \\ + \left[\bar{t}\ln\left(\frac{2\bar{t}+1}{2\bar{t}-1}\right) - 1\right]\Theta(\bar{t}-1),$$

two-level form factor,

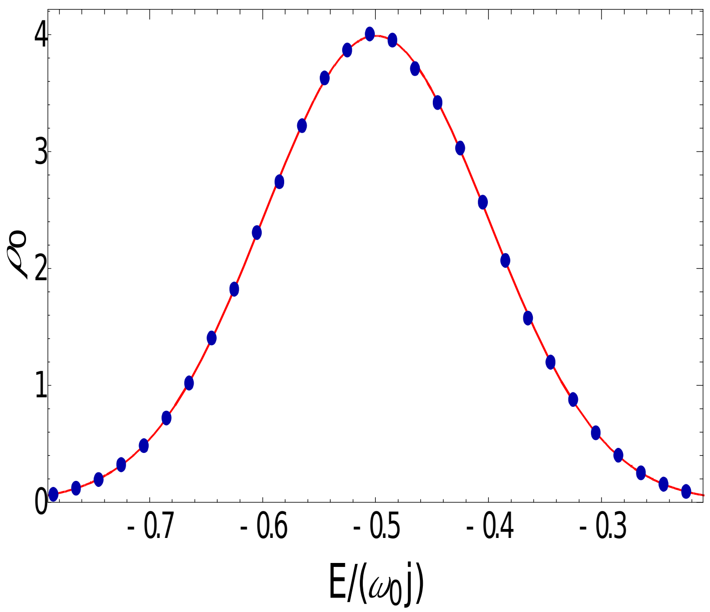
Equilibration and survival probability

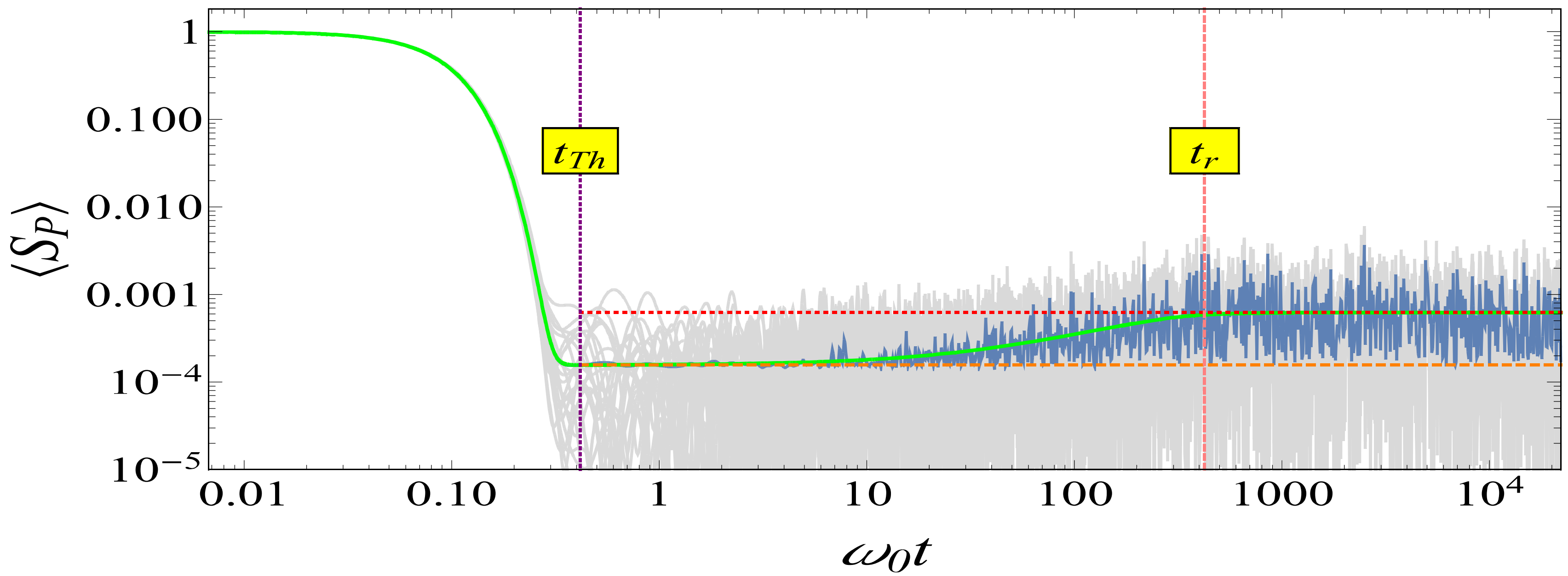
$$S_P(t) = |\langle\Psi(0)|\Psi(t)\rangle|^2 = S_P(t) = \left|\sum_k |c_k|^2 e^{-iE_k t}\right|^2$$





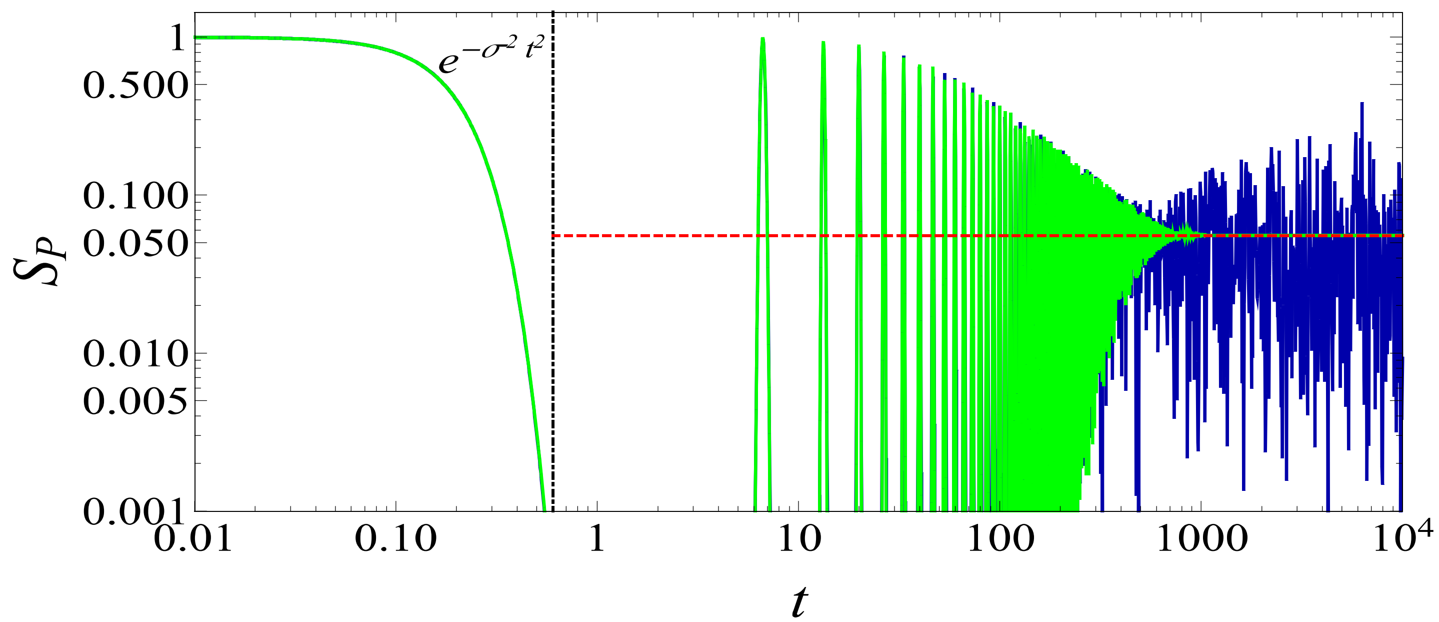
The correlation hole is a signature of quantum chaos





S. Lerma-Hernández, D. Villaseñor, M.A. Bastarrachea-Magnani, E. J. Torres-Herrera, L. F. Santos, and J. G. Hirsch, Phys. Rev. **E 100** (2019) 012218.

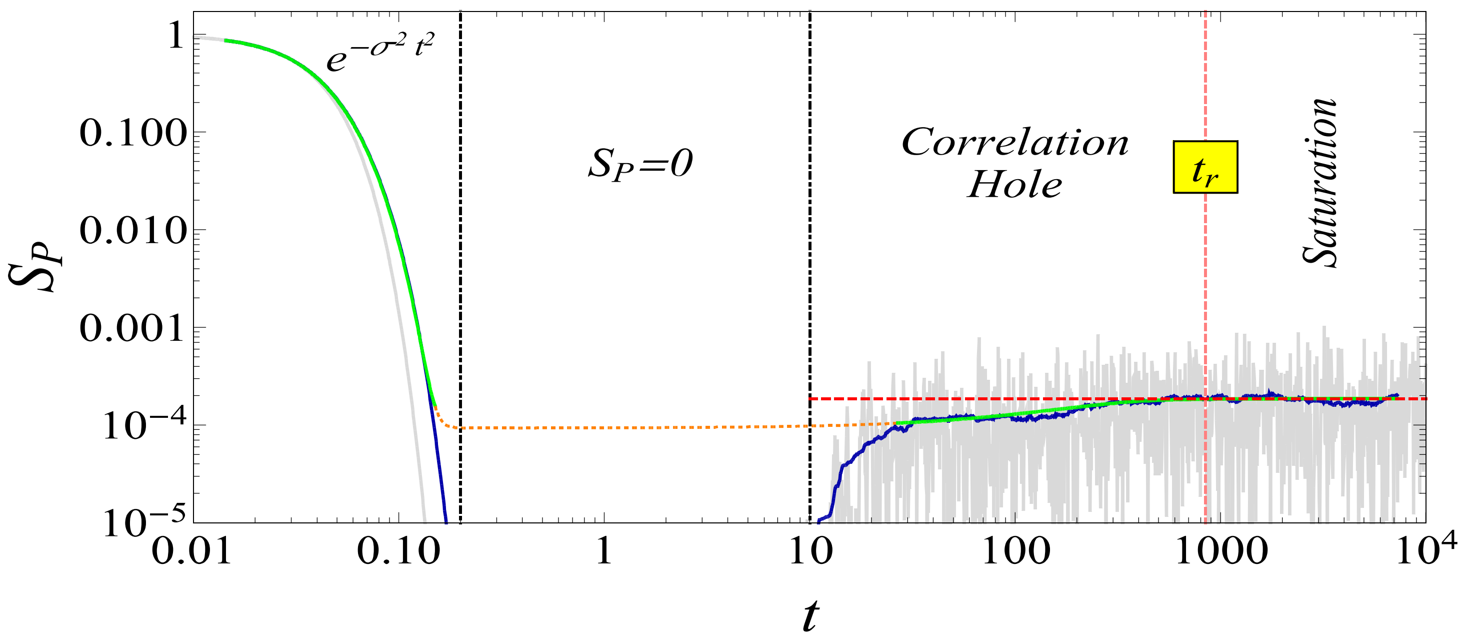
Survival probability of coherent states - quantum



Regular state:

* NO correlation hole

- Analytic description Sergio Lerma-Hernández, Jorge Chávez-Carlos, Miguel Angel Bastarrachea-Magnani, Lea F. Santos, and Jorge G. Hirsch. J. Phys. A: Math. Theor. **51** (2018) 475302



Chaotic state: correlation hole, BUT the survival probability is zero at short times

D. Villaseñor, S. Pilatowsky-Cameo, M. A. Bastarrachea-Magnani, S. Ler-ma, L. Santos, J.G.Hirsch
New Journal of Physics **22** (2020) 063036

Survival probability of coherent states – semiclassical

Fully chaotic region. D. Villaseñor, et al, New Journal of Physics 22 (2020) 063036.