Equivalent non-rational extensions of the harmonic oscillator, their ladder operators, and coherent states

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Equivalent non-rational extensions of the harmonic oscillator, their ladder operators, and coherent states

- Heisenberg-Weyl algebra
- Coherent states
- Supersymmetric quantum mechanics
- Equivalent non-rational extensions of the harmonic oscillator and ladder operators
- Coherent states and their properties
- Concluding remarks

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Heisenberg-Weyl algebra

Let us take as H the harmonic oscillator Hamiltonian,

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2,$$

whose eigenfunctions and eigenvalues are given by

$$\psi_n(x) = \sqrt{\frac{1}{2^n\sqrt{\pi}n!}}e^{-\frac{x^2}{2}}H_n(x), \quad E_n = n + \frac{1}{2}, \quad n = 0, 1, 2, \ldots,$$

where $H_n(x)$ is the Hermite polynomial of n-th degree. The first-order differential ladder operators

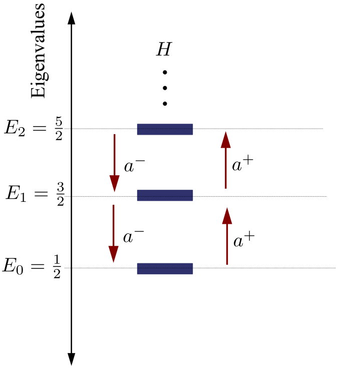
$$a^- = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right), \quad a^+ = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right),$$

factorize the oscillator Hamiltonian as $a^+a^- = H - \frac{1}{2}$ and $a^-a^+ = H + \frac{1}{2}$; they also generate the Lie algebra (Heisenberg-Weyl) $[H, a^{\pm}] = \pm a^{\pm}, \; [a^-, a^+] = \mathbb{I}$, which indicates that work as ladder operators.

Heisenberg-Weyl algebra

$$[H, a^+] = +a^+ \quad \rightarrow \quad Ha^+ = a^+(H+1)$$

$$[H, a^-] = -a^+ \quad \rightarrow \quad Ha^- = a^-(H-1)$$



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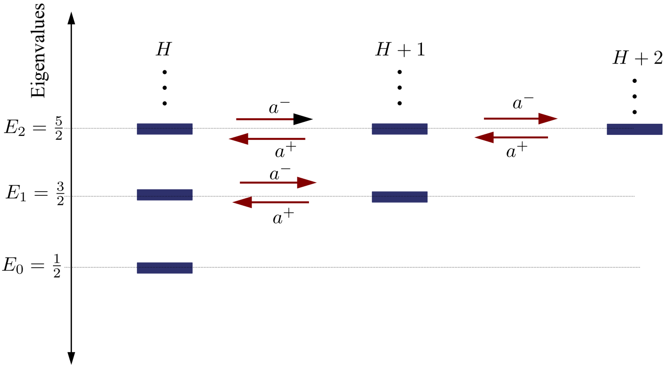


Coherent States

The coherent states of the harmonic oscillator are given by

$$|z\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle .$$

These states can be derived from four definitions:

1. Eigenstates of the annihilaton operator, $a^- |z\rangle = z |z\rangle$.
2. They are obtained by applying the displacement operator $D(z) \equiv e^{za^+ - z^* a^-}$ on the harmonic oscillator vacuum state, i.e. $|z\rangle = D(z) |0\rangle$.
3. They are states with minimum uncertainty relation for the position q and momentum p operators: $\langle \triangle q\rangle^2 \langle \triangle p\rangle^2 = \frac{1}{4}$.
4. They are an overcomplete set satisfying the completeness relation
$$\frac{1}{\pi} \int_{\mathbb{C}} |z\rangle \langle z| \, d^2 z = \mathbb{I}_{\mathbb{H}}.$$

Supersymmetric quantum mechanics

Let us suppose that H is an initial solvable Hamiltonian with eigenfunctions $\psi_n(x)$ and eigenvalues E_n, $n = 0, 1, 2, \ldots$. It is assumed the following relations

$$\widetilde{H}B^+ = B^+H, \quad HB = B\widetilde{H}.$$

where $H, \widetilde{H}$ are the two Schrödinger Hamiltonians

$$\widetilde{H} = -\frac{1}{2}\frac{d^2}{dx^2} + \widetilde{V}(x), \quad H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x).$$

We suppose as well that k solutions of the Schrödinger equation

$$Hu_j = \epsilon_j u_j, \quad j = 1,\ 2, \ldots, k,$$

for k different factorization energies ϵ_j are given. Then, the SUSY partner potential $\widetilde{V}$ will be given by

$$\widetilde{V} = V - [\ln W(u_1, u_2, \ldots, u_k)]''.$$

The eigenfunctions of the Hamiltonian $\widetilde{H}$ can be obtained via

$$\widetilde{\psi}_n \propto B^+\psi_n = \frac{1}{\sqrt{2^k}} \frac{W(u_1, u_2, \ldots, u_k, \psi_n)}{W(u_1, u_2, \ldots, u_k)},$$

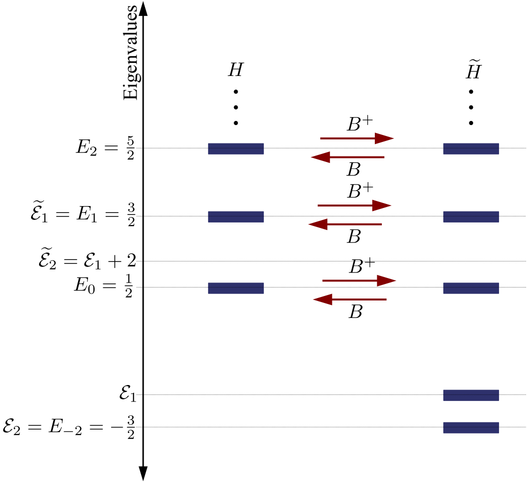
with eigenvalue E_n.
The SUSY partner Hamiltonian $\widetilde{H}$ could have a finite number of additional eigenfunctions $\widetilde{\psi}_{\epsilon_j}$, known as missing states, which corresponding eigenvalues are the factorization energies ϵ_j.
They can be written as

$$\widetilde{\psi}_{\epsilon_j} \propto \frac{W(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_k)}{W(u_1, u_2, \ldots, u_k)}.$$

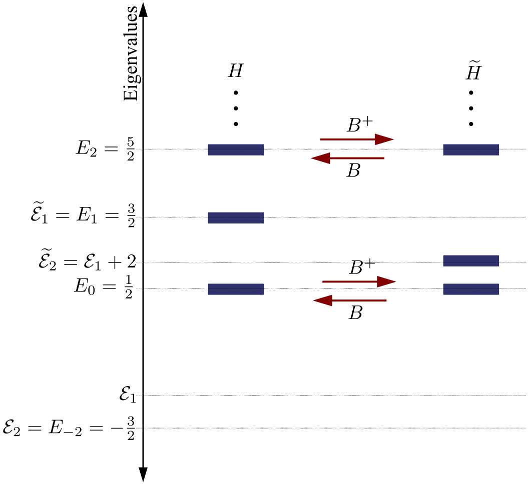
The intertwining operators B^+, B satisfy as well the following factorizations:

$$B^+B = (\widetilde{H} - \epsilon_1)\ldots(\widetilde{H} - \epsilon_k), \quad BB^+ = (H - \epsilon_1)\ldots(H - \epsilon_k).$$

Supersymmetric quantum mechanics



Supersymmetric quantum mechanics



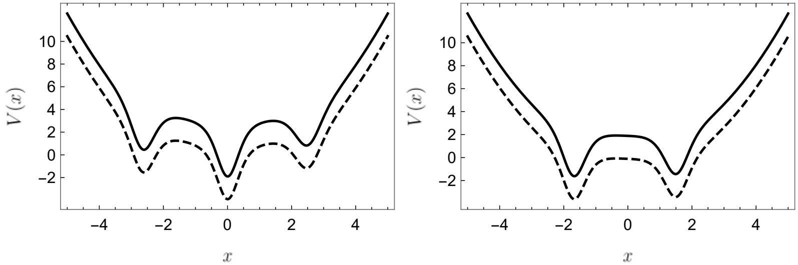
Equivalent non-rational extensions of the harmonic oscillator and ladder operators

Let us first write down the general solution of the stationary Schrödinger equation for the harmonic oscillator Hamiltonian with energy parameter $\mathcal{E} = \lambda + 1/2$, as follows

$$u(x) = e^{-\frac{x^2}{2}}[H_\lambda(x) + \gamma H_\lambda(-x)],$$

where γ is a real parameter.
Now, we will use two different BUT equivalent SUSY transformation to modify the harmonic oscillator Hamiltonian.

Equivalent non-rational extensions of the harmonic oscillator and ladder operators



Equivalent non-rational extensions of the harmonic oscillator

First SUSY transformation

Let us generate now a 2-SUSY partner potential $V^{(1)}$ of the harmonic oscillator. We employ the seed solutions

$$u_1^{(1)}(x) = e^{-\frac{x^2}{2}}[H_{\lambda_1}(x) + \gamma H_{\lambda_1}(-x)], \quad u_2^{(1)}(x) = \varphi_1(x),$$

with factorization energies $-3/2 < \mathcal{E}_1 < 1/2$, and $\mathcal{E}_2 = E_{-2} = -3/2$, respectively with $\lambda_1 = \mathcal{E}_1 - 1/2$. To obtain a nodeless Wronskian $W(u_1^{(1)}, u_2^{(1)})$ we must take $\gamma > 0$. The SUSY partner potential $V^{(1)}$ is

$$V^{(1)} = \frac{x^2}{2} - [\ln W(u_1^{(1)}, u_2^{(1)})]''.$$

The intertwining operators $B^{(1)}, B^{(1)+}$ relate the Hamiltonian $H^{(1)}$ with the oscillator Hamiltonian in the way

$$H^{(1)} B^{(1)+} = B^{(1)+} H, \quad HB^{(1)} = B^{(1)} H^{(1)}.$$

Second SUSY transformation

We can obtain basically the same spectrum by deleting the first excited level and adding a new level at the right place. We will employ now the seed solutions

$$u_1^{(2)}(x) = \psi_1(x), \quad u_2^{(2)}(x) = e^{-\frac{x^2}{2}}[H_{\lambda_2}(x) + \gamma H_{\lambda_2}(-x)]$$

with factorization energies $\widetilde{\mathcal{E}}_1 = E_1$ and $\widetilde{\mathcal{E}}_2 = \mathcal{E}_1 + 2$, respectively, thus $E_0 < \widetilde{\mathcal{E}}_2 < E_2$, and $\lambda_2 = \lambda_1 + 2$. Moreover, we will take the same value of γ as in the previous SUSY transformation. The Wronskian $W(u_1^{(2)}, u_2^{(2)})$ can be written as

$$W(u_1^{(2)}, u_2^{(2)}) = e^{-x^2}\mathcal{W}(x),$$

The SUSY partner potential can be expressed as

$$V^{(2)} = \frac{1}{2}x^2 + 2 - [\ln \mathcal{W}(x)]''.$$

Again, $B^{(2)}$, $B^{(2)+}$ intertwine the Hamiltonian $H^{(2)}$ with H as

$$H^{(2)}B^{(2)+} = B^{(2)+}H, \qquad HB^{(2)} = B^{(2)}H^{(2)}.$$

Ladder Operators

Let us define now the operators:

$$\mathcal{L}^+ = B^{(1)+}B^{(2)}, \quad \mathcal{L}^- = B^{(2)+}B^{(1)}.$$

Since these operators fulfill the following commutation relations

$$[\widetilde{H}, \mathcal{L}^\pm] = \pm 2\mathcal{L}^\pm,$$
$$[\mathcal{L}^-, \mathcal{L}^+] = (\widetilde{H} + 2 - \mathcal{E}_1)(\widetilde{H} + 2 - \mathcal{E}_2)(\widetilde{H} + 2 - \widetilde{\mathcal{E}}_1)(\widetilde{H} + 2 - \widetilde{\mathcal{E}}_2)$$
$$- (\widetilde{H} - \mathcal{E}_1)(\widetilde{H} - \mathcal{E}_2)(\widetilde{H} - \widetilde{\mathcal{E}}_1)(\widetilde{H} - \widetilde{\mathcal{E}}_2),$$

they are in fact ladder operators that connect eigenstates whose energy levels differ by two energy units of $\widetilde{H}$.
The operators $\widetilde{H}$, $\mathcal{L}^-$, $\mathcal{L}^+$ realize a polynomial Heisenberg algebra of third degree.
The kernel of the annihilation operator $\mathcal{L}^-$ is generated by the functions

$$K_{\mathcal{L}^-} = \{\widetilde{\psi}_{E_{-2}}, \widetilde{\psi}_\epsilon, \widetilde{\psi}_1, B^{(1)+}u_2^{(2)}\}.$$

Equivalent non-rational extensions of the harmonic oscillator and ladder operators

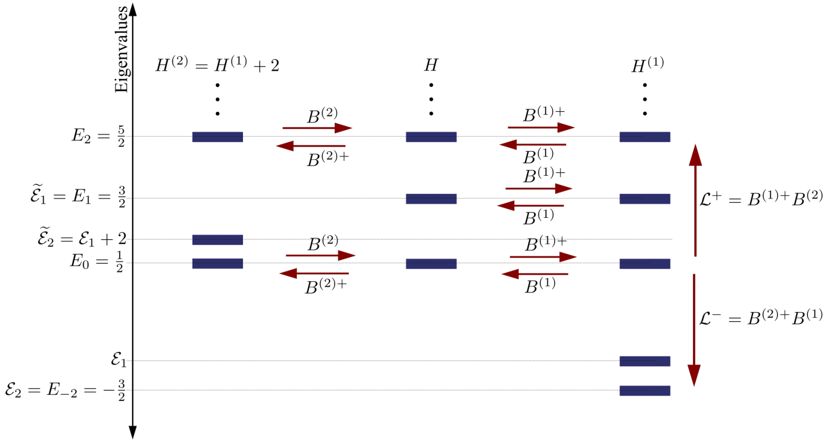


Diagram of the action of the two-step ladder operators $\mathcal{L}^-, \mathcal{L}^+$.

Coherent states and their properties

Let us construct the Barut-Girardello coherent states in the standard way,

$$\mathcal{L}^- |z\rangle = z\, |z\rangle \,,$$

Recall that $[\widetilde{H}, \mathcal{L}^\pm] = \pm 2\mathcal{L}^\pm$, thus the Hilbert space $\mathbb{H}$ can be decomposed as the direct sum of two infinite dimension subspaces (labelled by the index $\nu = -2,\ 1$, i.e., $\mathbb{H}^{-2}$ and $\mathbb{H}^{1}$, respectively) plus the subspace $\mathbb{H}^{\epsilon}$ spanned by the single eigenstate $\widetilde{\psi}_\epsilon$.
After some math, the coherent states can be written as

$$|z^\nu\rangle = c_0 \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n \sqrt{\frac{\Gamma\left(\frac{2\nu-2\epsilon+5}{4}\right)\Gamma\left(\frac{2\nu-2\epsilon+1}{4}\right)\Gamma\left(\frac{\nu+4}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{2\nu-2\epsilon+5}{4}+n\right)\Gamma\left(\frac{2\nu-2\epsilon+1}{4}+n\right)\Gamma\left(\frac{\nu+4}{2}+n\right)\Gamma\left(\frac{\nu+1}{2}+n\right)}}\, |\nu+2n\rangle \,,$$

where c_0 is the normalization constant given by

$$c_0 = \left[{}_1F_4\left(1; \frac{2\nu-2\epsilon+5}{4}, \frac{2\nu-2\epsilon+1}{4}, \frac{\nu+4}{2}, \frac{\nu+1}{2}; \frac{|z|^2}{16}\right)\right]^{-1/2} .$$

Coherent states and their properties

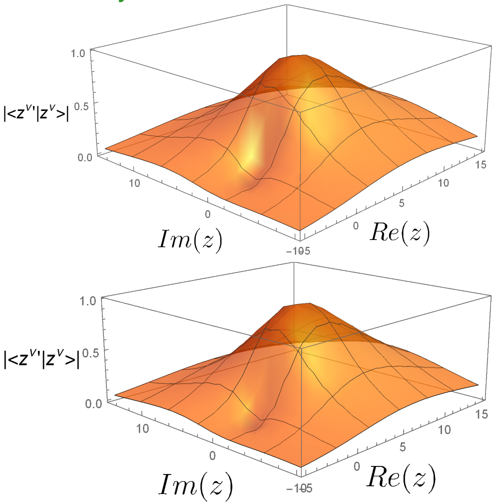
- Completeness relation
  We can show that our coherent states they fulfill a completeness relation of the form

$$\int_{\mathbb{C}} |z^\nu\rangle \langle z^\nu| \mu(z) d^2 z = \mathbb{I}_{\mathbb{H}^\nu}$$

  on each Hilbert subspace $\mathbb{H}^\nu$ spanned by the kets $|\nu + 2n\rangle$, where $\mu(z)$ is a positive definite measure function.

Coherent states and their properties

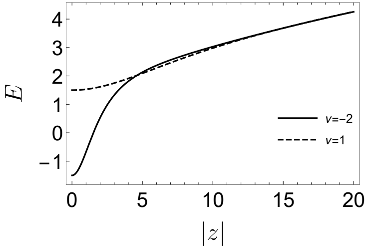
- Continuity on the label

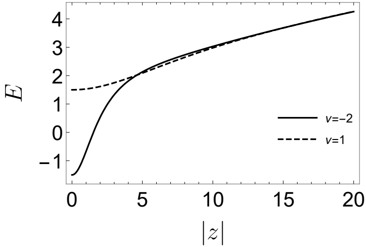


Modulus of the scalar product $\left\langle z'^{\nu} \middle| z^{\nu} \right\rangle$ of the coherent states $|z^{\nu}\rangle$ and $\left|z'^{\nu}\right\rangle$ for $z' = 4 + i$ in the subspace with $\nu = -2$ (**up**), and $\nu = 1$ (**down**).

Coherent states and their properties

- Mean energy values





Mean energy values as function of $|z|$ in both subspaces for $\epsilon = -\frac{1}{4}$ (**up**), and $\epsilon = -\frac{3}{4}$ (**down**).

Coherent states and their properties

- Temporal stability

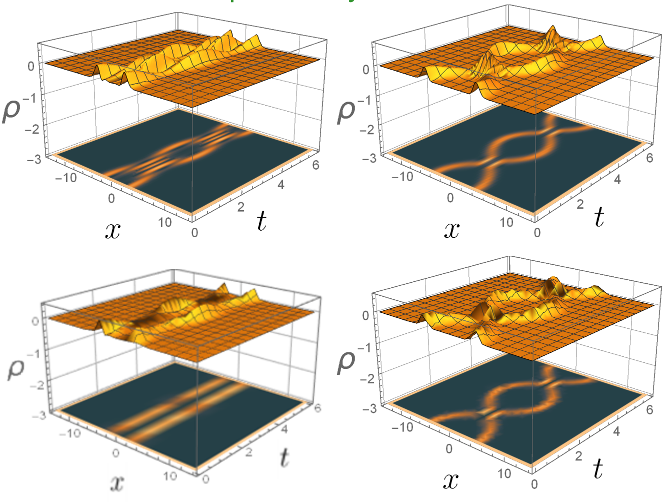
  By applying the time evolution operator, $U(t) = e^{-i\widetilde{H}t}$, onto our coherent states, it is found the corresponding temporal evolution

$$U(t)\,|z^{\nu}\rangle = e^{-iE_{0,\nu}t}\left|\left(ze^{-i2t}\right)^{\nu}\right\rangle,$$

  where $E_{0,\nu} = \nu + 1/2$. Up to global phase factor, they evolve in time into another coherent states in the same subspace. As a first indication of their non-classical behavior, it can be seen that the period of every cyclic evolution is $\tau = \pi$, half the period of the standard coherent states.

Coherent states and their properties

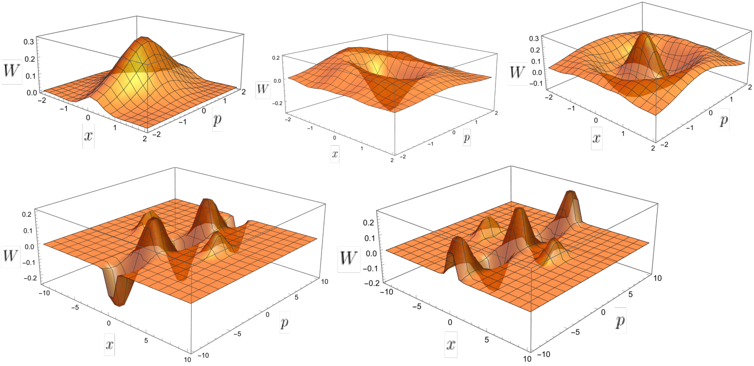
- Time evolution of probability densities



Probability densities of the coherent states with $\epsilon = 0$, $\gamma = 2$. **First:** $\nu = -2$, $z = 10$; **Second:** $\nu = -2$, $z = 100$; **Third:** $\nu = 1$, $z = 10$; **Fourth:** $\nu = 1$, $z = 100$.

Coherent states and their properties

- ## Wigner distributions



Wigner functions of the ground state $\widetilde{\psi}_{E_{-2}}$ (**top left**), the first excited state $\widetilde{\psi}_{\epsilon}$ (**top center**), the second excited state $\widetilde{\psi}_0$ (**top left**), coherent state with $\nu = -2$, $z = 100$ (**bottom left**), and the coherent state with $\nu = 1$, $z = 100$ (**bottom right**). The parameters employed are $\epsilon = 0$, $\gamma = 2$.

Summary

- We demonstrated that non-rational SUSY extensions of the harmonic oscillator have equivalent partners.
- Using this equivalence, we were able to find novel ladder operator for such extensions.
- We constructed families of coherent states as eigenstates of the annihilation operator and studies its properties.

Thanks