CSIC
CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS

IEM
Instituto de
Estructura
de la Materia

Estudiando la transición entre integrabilidad y caos en sistemas cuánticos abiertos

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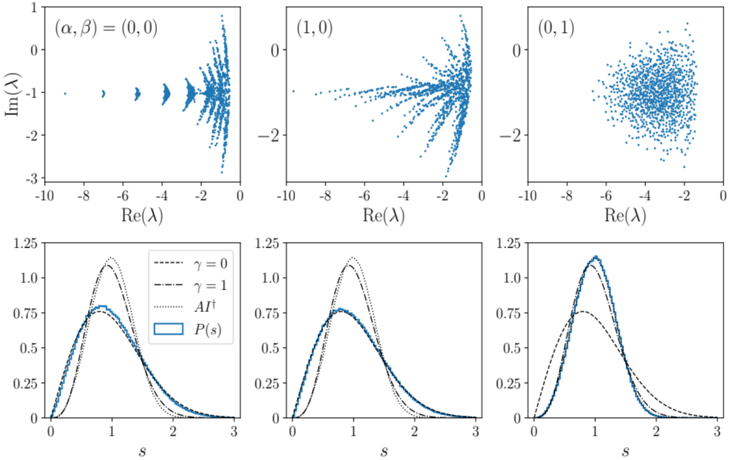

arXiv:2102.13452

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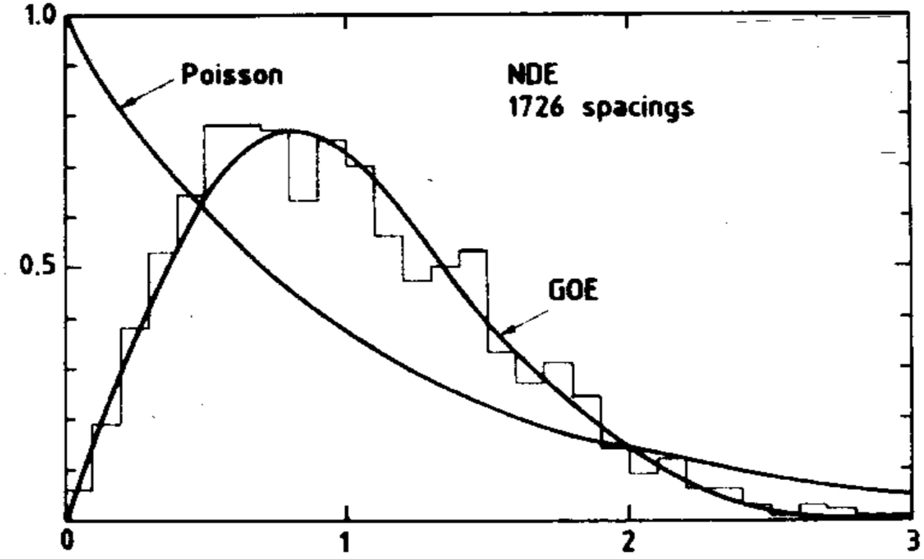
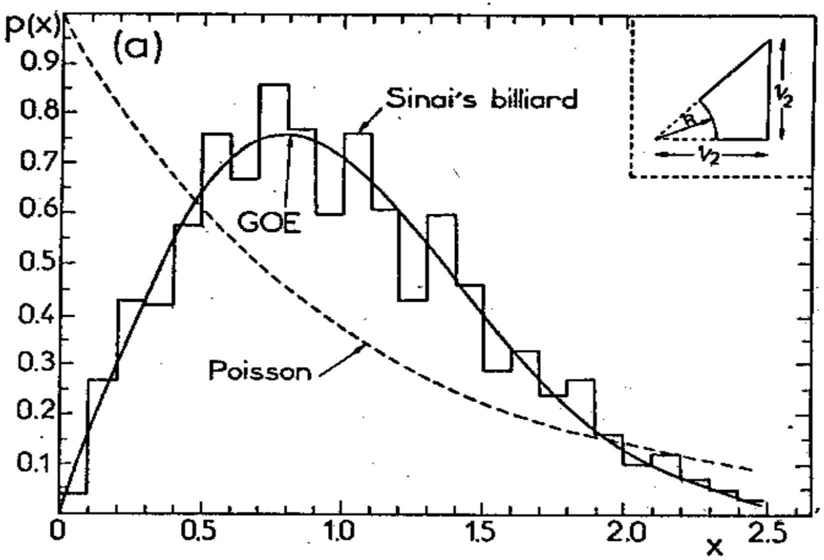
Quantum chaos



Bohigas, Haq, Pandey, in Nuclear Data for Science and Technology,
K.H. Böchhoff (Ed.) (Reidel, Dordrecht, 1983) p. 809



Bohigas, Giannoni, Schmidt PRL 52, 1 (1984)

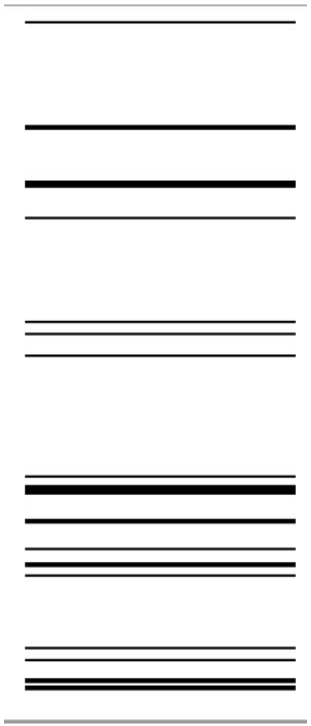
Universal statistical properties

$$s = \frac{E_{i+1} - E_i}{\bar{\rho}}$$

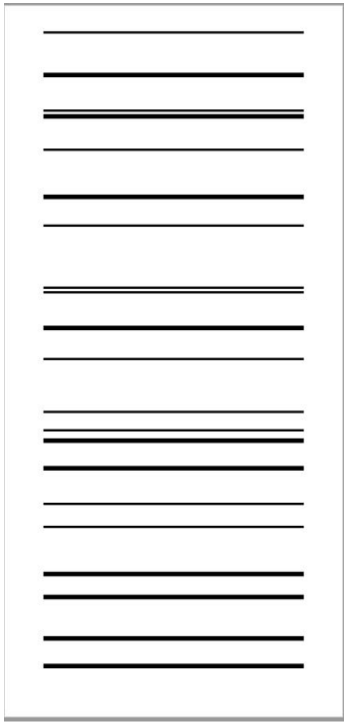
$$P_W(S) = \frac{\pi S}{2} \exp\left(-\frac{\pi S^2}{4}\right)$$

Spectral statistics

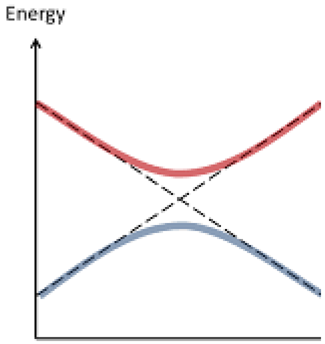
Espectro regular



Espectro caótico



Level repulsion



Integrable systems

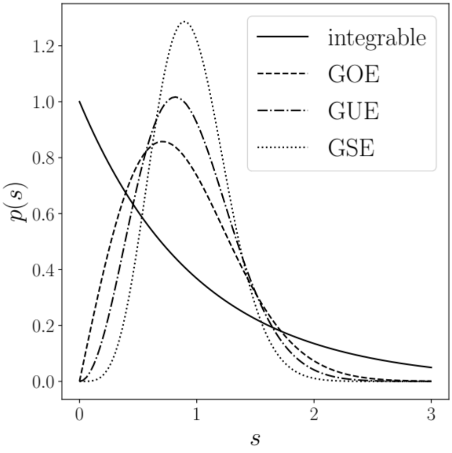
$$P(s) = e^{-s}$$

Chaotic systems

$$P(s) \propto s^{\beta} \qquad \beta = 1, 2, 4$$

Spectral statistics

- Gaussian Orthogonal Ensemble (GOE) β=1 (Time reversal symmetry and rotational symmetry)
- Gaussian Unitary Ensemble (GUE) β=2 (Broken time reversal symmetry)
- Gaussian Symplectic Ensembe (GSE) β=4 (Time reversal symmetry but no rotational symmetry)

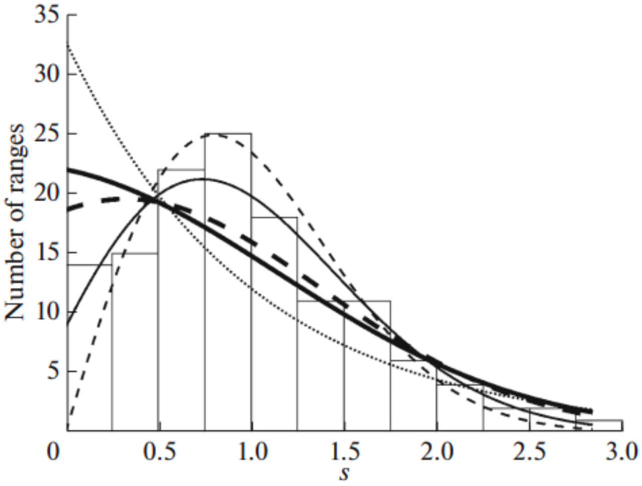


Wigner surmise: 2x2 matrices

$$P(s) = A(\beta)s^{\beta}e^{-B(\beta)s^2}$$

Intermediate systems: Non-universal spectral statistics

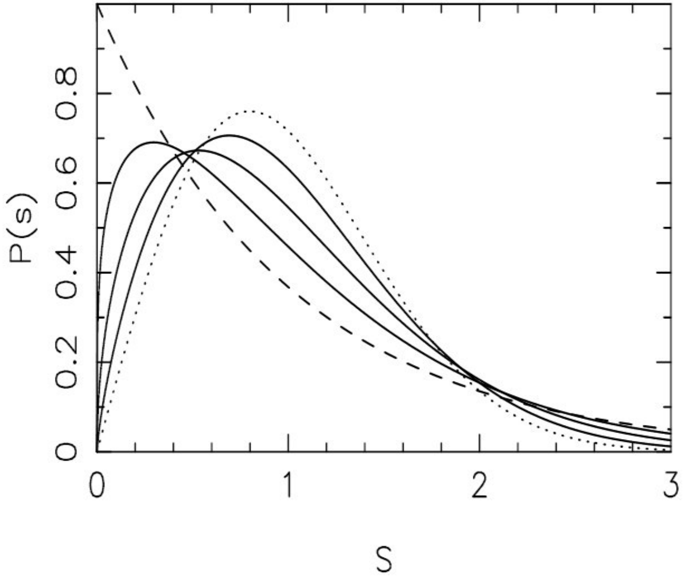
Berry-Robnik distribution



Syshchenko et al. Poverknnosk 6, 103 (2020)

$$P(0) = F$$

Brody distribution



$$P_\beta(s) = \alpha(\beta+1)\, s^\beta \exp\left(-\alpha s^{\beta+1}\right) \quad \alpha = \left(\Gamma\left[\frac{\beta+2}{\beta+1}\right]\right)^{\beta+1}$$

Integrable systems and Richardson-Gaudin models

- Quantum integrability (under discussion): Every eigenstate and eigenvalue can be found with algebraic complexity.
- Richardson-Gaudin models defined by the integrals of motion.

$$R_i = S_i^z - g \sum_{j \neq i} \frac{X_{ij}}{2} \left(S_i^+ S_j^- + S_i^- S_j^+ \right) + Z_{ij} S_i^z S_j^z, \quad [R_i,\, R_j] = 0$$

- Hamiltonian: linear combination

$$H = \sum_i \varepsilon_i R_i, \quad [H,\, R_i] = 0$$

Rational: $X_{ij} = Z_{ij} = \frac{1}{\eta_i - \eta_j}$ Hyperbolic: $X_{ij} = 2\frac{\sqrt{\eta_i \eta_j}}{\eta_i - \eta_j}, \quad Z_{ij} = \frac{\eta_i \eta_j}{\eta_i - \eta_j}$

Rational Richardson-Gaudin model

- Exact eigenstates:

$$|\Psi\rangle = \prod_{\alpha=1}^{M} \left(\sum_{i} \frac{1}{\eta_i - \underset{\uparrow}{E_\alpha}} S_i^{+} \right) |\nu\rangle$$

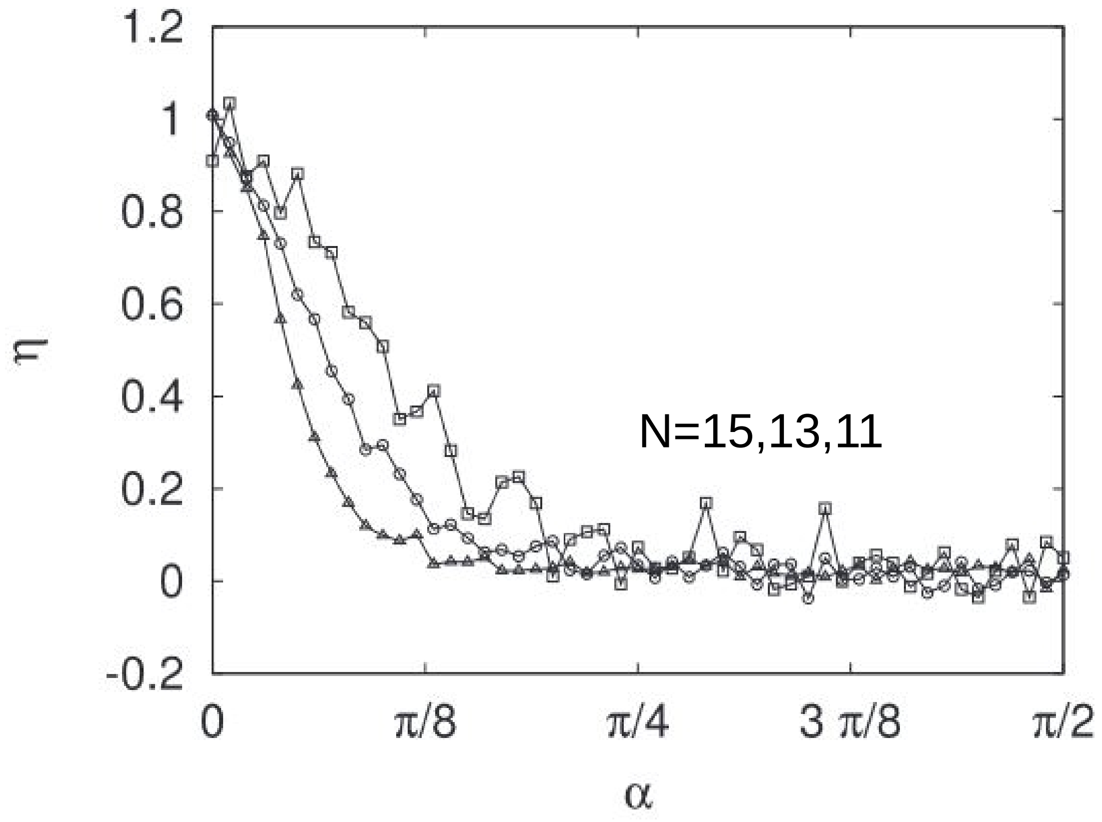
Pair energies

- Finding the pair energies → Richardson-Gaudin equations

$$1 + \frac{g}{2} \sum_{i=1}^{L} \frac{1}{\eta_i - E_\alpha} + g \sum_{\beta(\neq\alpha)}^{M} \frac{1}{E_\alpha - E_\beta} = 0$$

Transition to chaos

$$\eta = \frac{\int_0^{s_0} \mathrm{d}s\,(P(s) - P_{\mathrm{Wigner}}(s))}{\int_0^{s_0} \mathrm{d}s\,(P_{\mathrm{Poisson}}(s) - P_{\mathrm{Wigner}}(s))}$$



A. Relaño J. Stat. Mech. P07016 (2010)

Open quantum systems: Lindblad equation

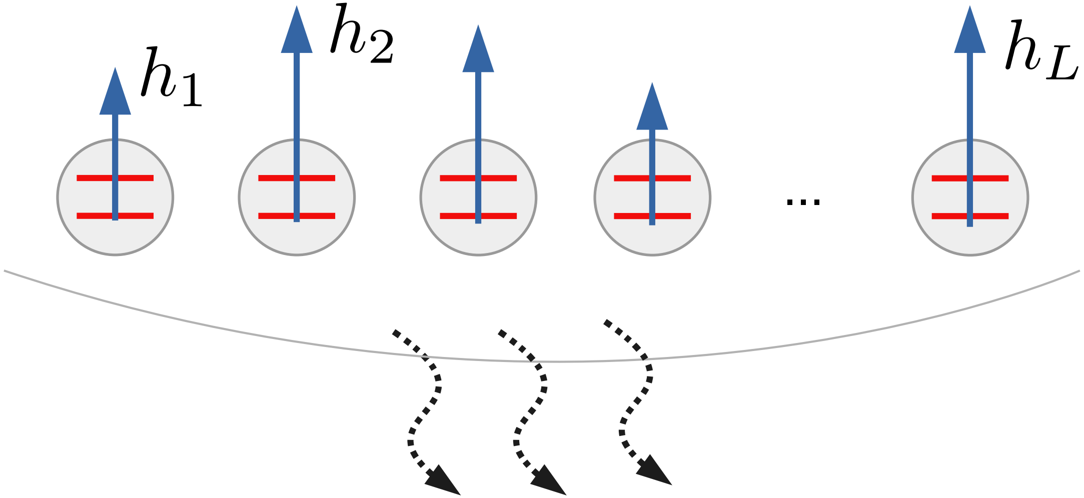
$$\partial_t \rho = \mathcal{L}(\rho) = -i\,[H,\rho] + \sum_j \left(L_j \rho L_j^\dagger - \frac{1}{2}\rho L_j^\dagger L_j - \frac{1}{2} L_j^\dagger L_j \rho \right)$$

Time evolution of an eigenstate: $\mathcal{L}(\rho_i) = \lambda_i \rho_i \qquad e^{\mathcal{L}t}\rho_i = e^{\lambda_i t}\rho_i$

Steady state (zero eigenvalue): $\lim_{t\to\infty} e^{\mathcal{L}t}\rho = \rho_{\mathrm{SS}}$

Simple spin chain Liouvillian

$$H = \sum_{i=1}^{L} h_i S_i^z$$



Dissipation rate

$$L_z = \sqrt{\Gamma_0} \sum_{i=1}^{L} z_i S_i^z, \quad L_\pm = \sqrt{\Gamma} \sum_{i=1}^{L} x_i S_i^\pm$$

Simple spin chain Liouvillian

- Vectorize space

$$\rho = \sum_{ij} c_{ij} |i\rangle\langle j| \rightarrow |\rho\rangle\rangle = \sum_{ij} c_{ij} |i\rangle \otimes |j\rangle$$

$$A\rho B \rightarrow A \otimes B^T |\rho\rangle\rangle, \quad S^\alpha := S^\alpha \otimes \mathbb{I}, \quad J^\alpha := \mathbb{I} \otimes S^\alpha$$

- Full Liouvillian

$$\mathcal{L} = -i \sum_i h_i K_i^z - \Gamma_0 \sum_{ij} z_i z_j K_i^z K_j^z - \frac{\Gamma}{2} \sum_{ij} x_i x_j \left(K_i^+ K_j^- + K_i^- K_j^+ \right)$$

$$K^\alpha = S^\alpha + J^\alpha \qquad [\mathcal{L}, K^z] = 0$$

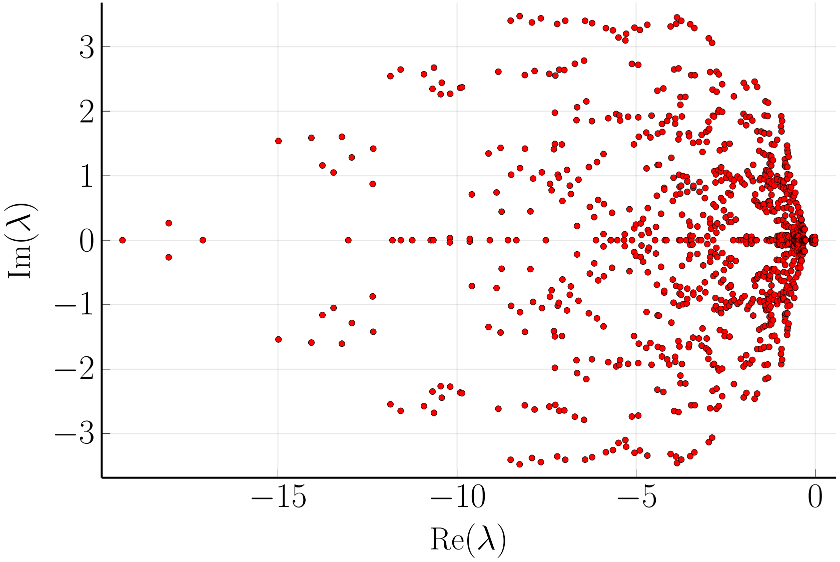
The Liouvillian superoperator

- Properties:
  - Linear operator → spectrum $\mathcal{L}(\rho_i) = \lambda_i \rho_i, \quad \rho_i(t) = e^{\lambda_i (t-t_0)} \rho_i(t_0)$
  - Real negative $\mathrm{Re}\{\lambda_i\} \leq 0$
  - Non-hermitian $\lambda_i \in \mathbb{C}$
  - Complex conjugate pairs

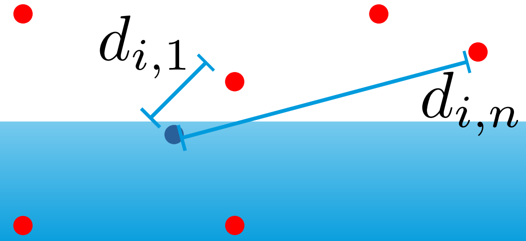
$$(\lambda_i,\, \rho_i) = (\lambda_i^*,\, \rho_i^\dagger)$$

  - Always a **steady state** $\lambda_{\mathrm{SS}} = 0$

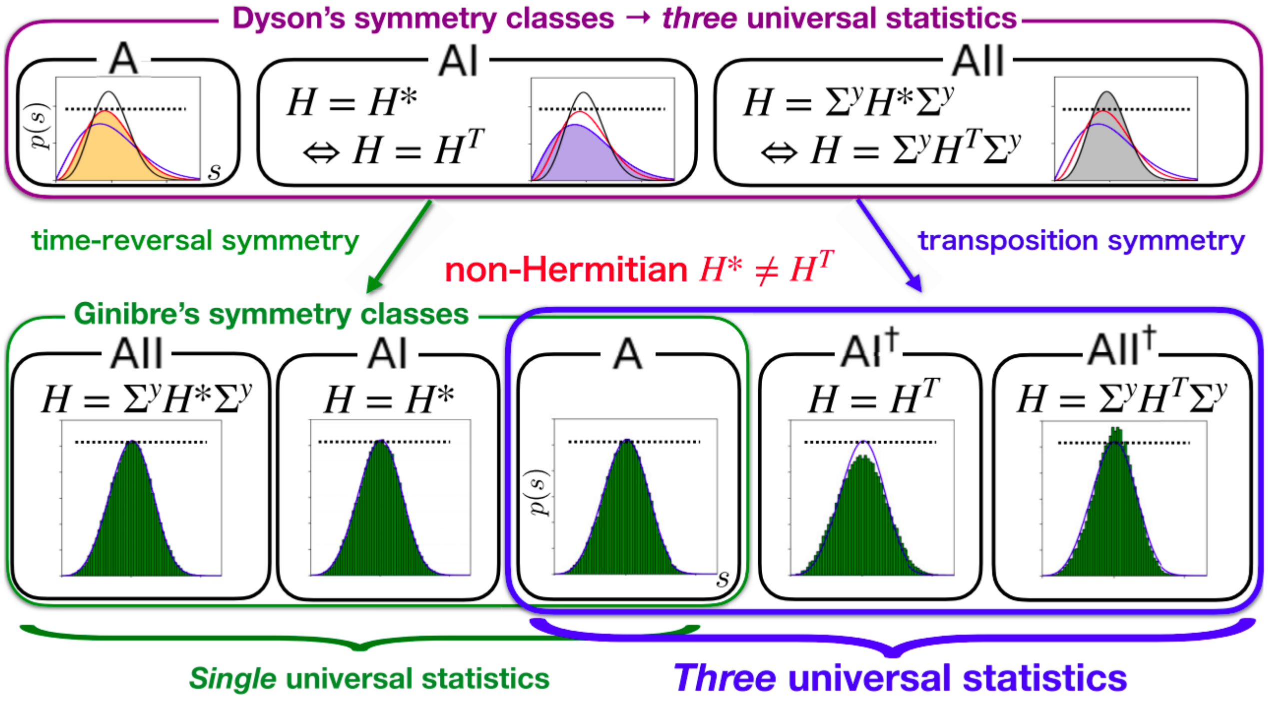
$$\lim_{t\to\infty} \rho(t) = \rho_{\mathrm{SS}}$$

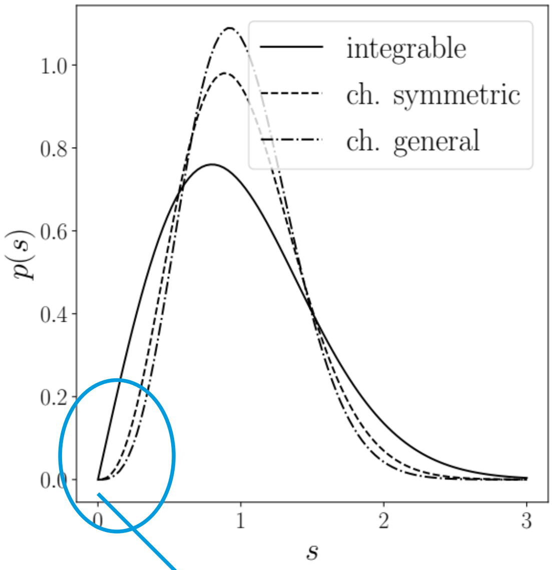


$$S_i = \sqrt{\frac{n}{\pi d_{i,n}^2}} d_{i,1}$$



Ginibre ensembles





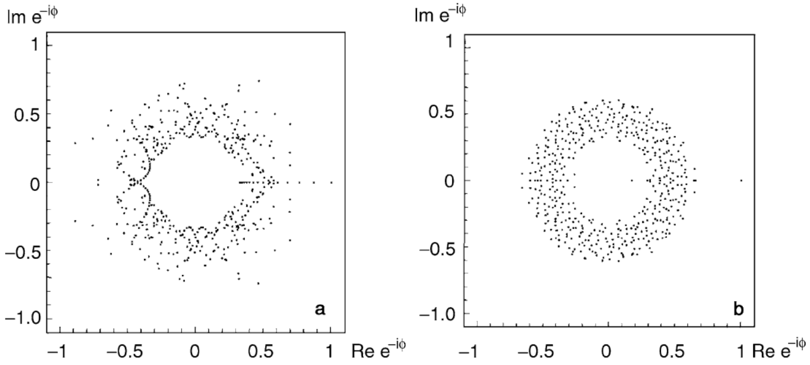
$P(s) \sim s^3$

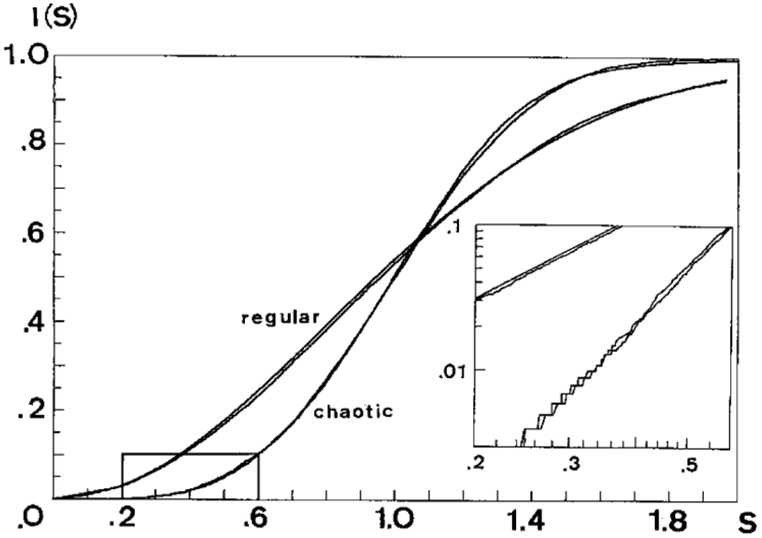
Hamazaki et al. Phys. Rev. Res. 2, 023286 (2020)

Integrable: 2D Poisson process $P(s) = \frac{\pi}{2} s e^{-\frac{\pi}{4}s^2}$

Chaos in open quantum systems

Integrable rotor plus weak dissipation vs.
Chaotic rotor plus weak dissipation





$$I(s) = \int_{s}^{0} P(s')ds'$$

R. Grobe, F. Haake, H.-J. Sommers, Phys. Rev. Lett. 61, 1899 (1988)

Making an integrable Liouvillian

- Richardson-Gaudin models can make integrable Liouvillians

$$R_j = -\underline{i}S_j^z - g\sum_{k\neq j}\frac{X_{jk}}{2}\left(S_j^+S_k^- + S_j^-S_k^+\right) + Z_{jk}S_j^zS_k^z$$

Doesn’t change integrability properties

$$\mathcal{L} = \sum_i \varepsilon_i R_i \qquad [\mathcal{L}, R_i] = [R_i, R_j] = 0$$

Making an integrable Liouvillian

- Find suitable system and jumps

Transition parameter

- $$H = \sum_i \eta_i S_i^z \qquad L_z^1 = \sqrt{(1-\alpha)\frac{g}{2}} \sum_i S_i^z, \quad L_z^2 = \sqrt{\alpha \frac{g}{2}} \sum_i \eta_i S_i^z$$
  $$L_\pm = \sqrt{\frac{g}{2}} \sum_i x_i S_i^\pm, \quad x_i = \sqrt{(1-\alpha) + \alpha \eta_i^2}$$

- Integrable Liouvillian: $\alpha = 0, 1$ rational (XXX), $\alpha \neq 0, 1$ hyperbolic (XXZ)

$$\mathcal{L}_{\mathrm{int}}(\alpha) = -i \sum_i \eta_i K_i^z - \frac{g}{4} \sum_{i \neq j} x_i x_j \left(K_i^+ K_j^- + K_i^- K_j^+ \right)$$
$$- \frac{g}{2} \sum_{i \neq j} \left[1 - \alpha + \alpha \eta_i \eta_j \right] K_i^z K_j^z$$

Chaotic Liouvillian

- Build chaotic Liouvillian with several random jumps

$$H = \sum_i \eta_i S_i^z \qquad L_\pm^a = \sqrt{g} \sum_i w_i^a S_i^\pm, \quad w_i^a \in \mathrm{unif}(-1/L, 1/L)$$

$$\mathcal{L}_{\mathrm{chaotic}} = -i \sum_i \eta_i K_i^z - \frac{g}{2} \sum_{a=1}^{n_j} \sum_{ij} w_i^a w_j^a \left(K_i^+ K_j^- + K_i^- K_j^+ \right)$$

Transition Liouvillian

- Simple interpolation with one variable
  - Integrable $\beta = 0$, still many integrable models with α
  - Chaotic $\beta = 1$

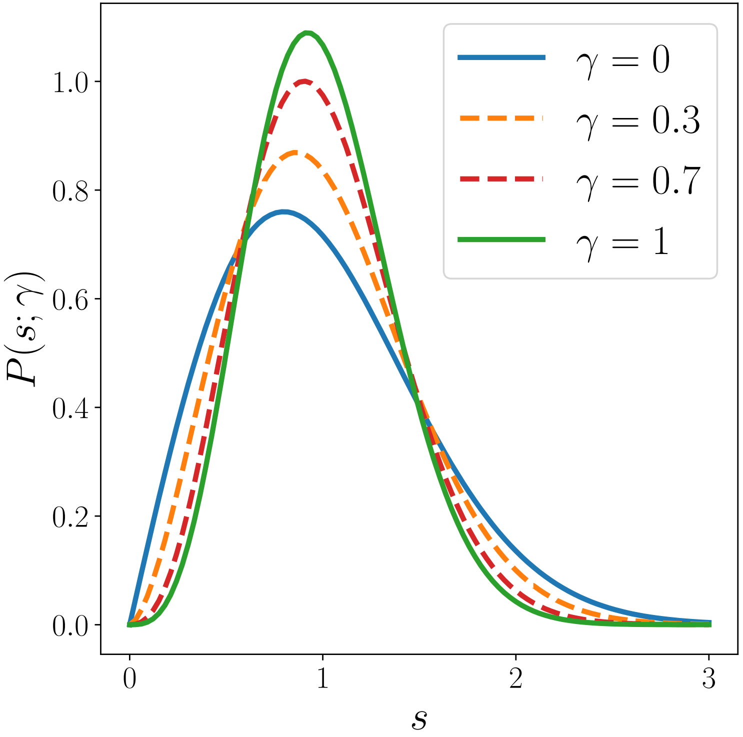
$$\mathcal{L}(\alpha, \beta) = (1 - \beta)\mathcal{L}_{\mathrm{int}}(\alpha) + \beta\mathcal{L}_{\mathrm{chaotic}}$$

Measuring the transition to chaos

$$P(s;\gamma) = A(\gamma)s^{2\gamma+1}e^{-B(\gamma)s^2}, \quad \gamma \in [0,1]$$

- 2d Poisson for $\gamma = 0$ → integrable

- Non hermitian Random Matrix $\gamma = 1$(not exact, but very close) → chaos



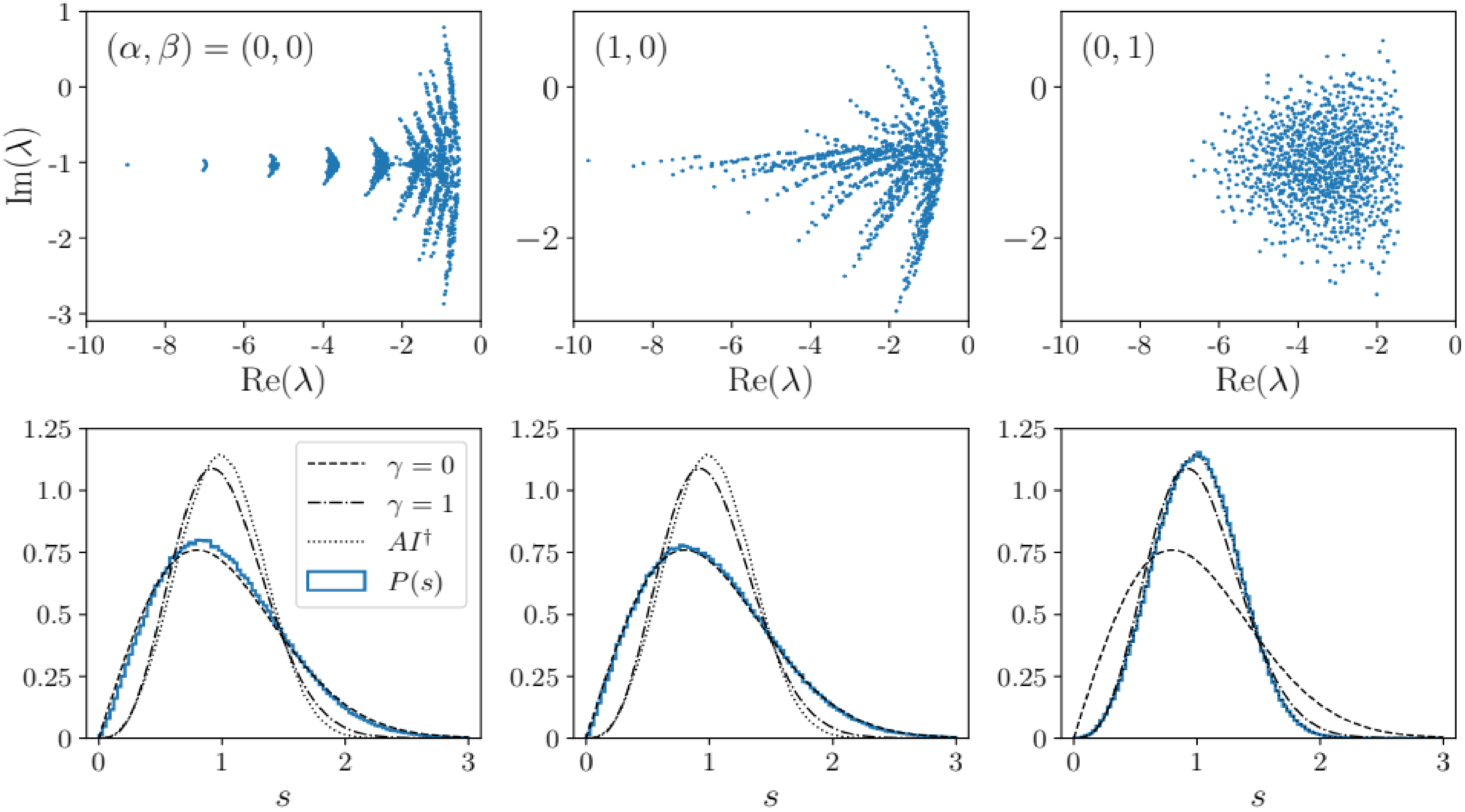
Transition to chaos

$n_j = L/2 = 3$

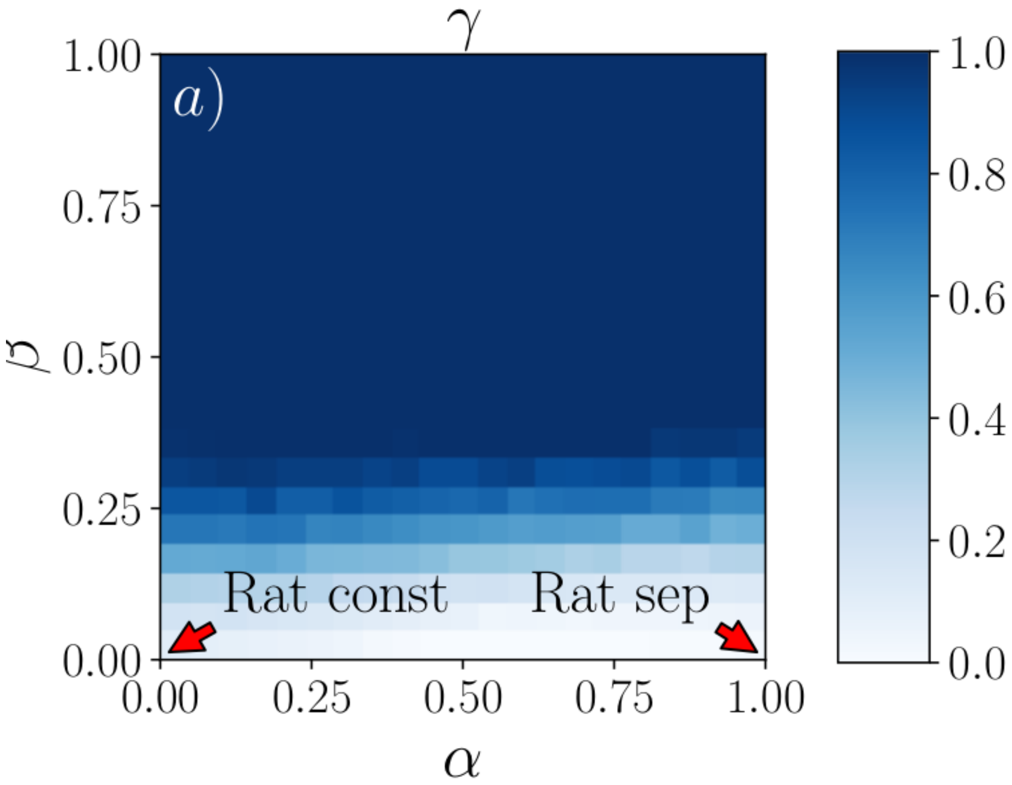
$\omega_j^a \in [-1/2, 1/2]$

$$\eta_i = 1/2 + i/L + \xi_i$$

$\xi_i \in [-1/L, 1/L]$

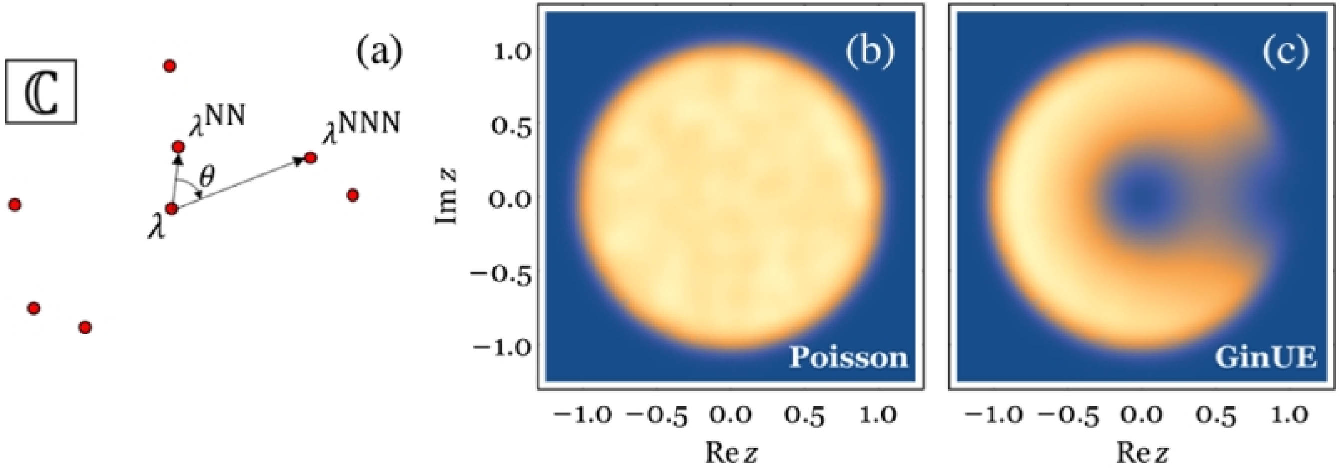


Transition to chaos P(s)



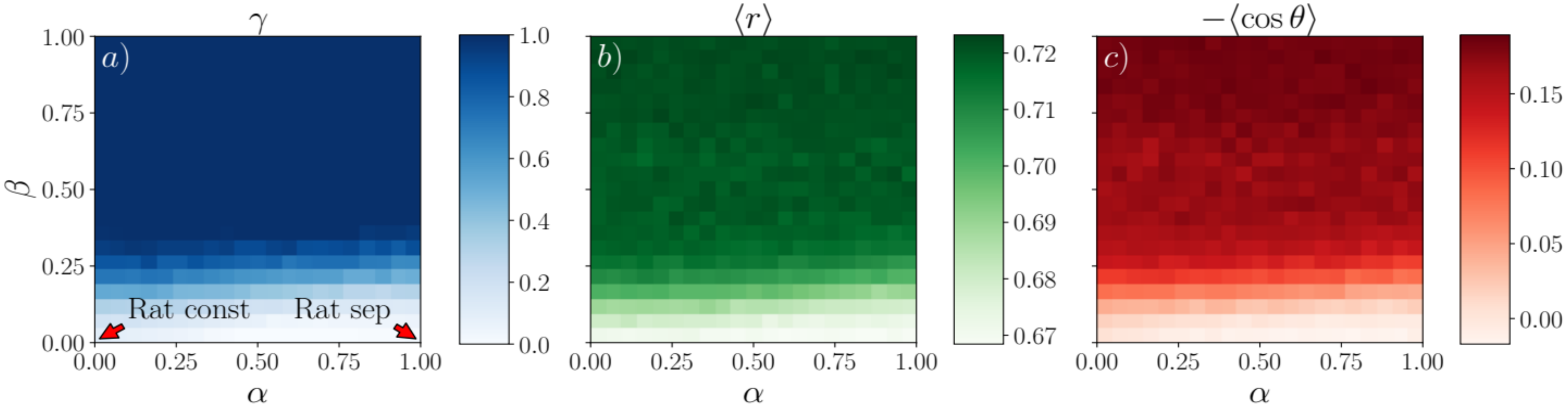
Complex ratios

$$z_k = \frac{\lambda_k^{\mathrm{NN}} - \lambda_k}{\lambda_k^{\mathrm{NNN}} - \lambda_k}$$

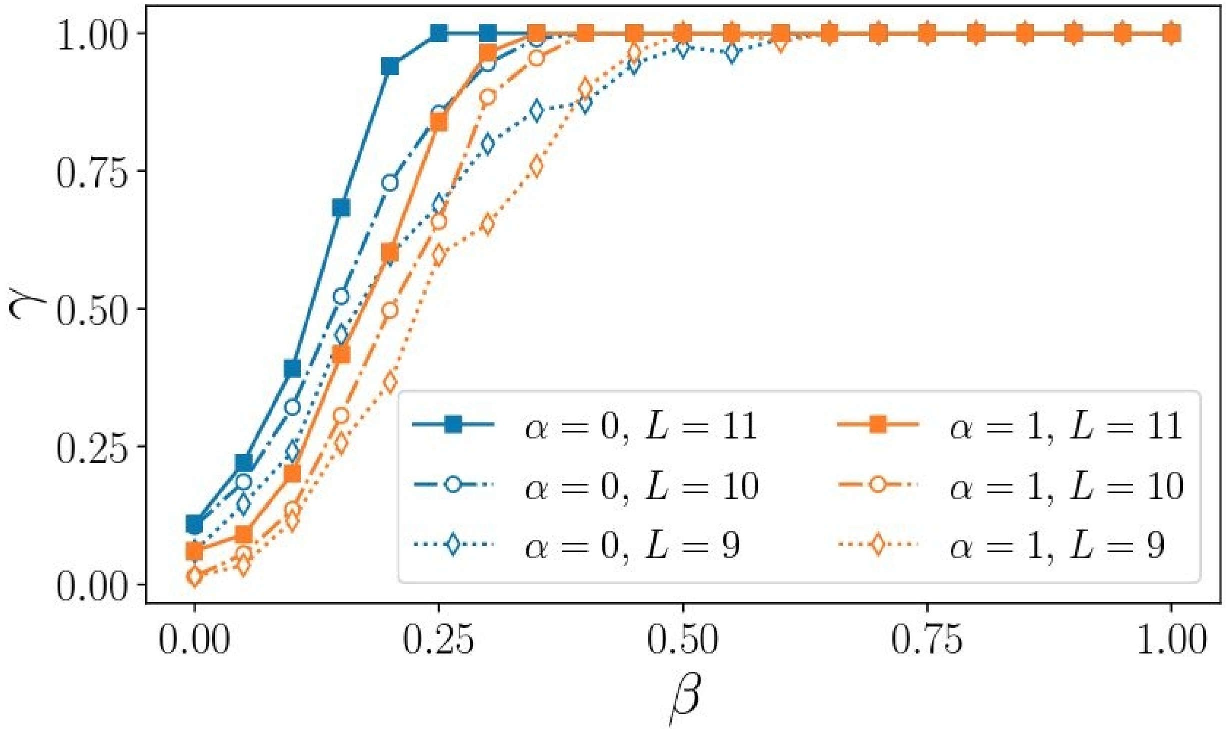


L. Sá, P. Ribeiro and T. Prosen, Phys. Rev. X 10, 021019 (2020)

Transition to chaos: level statistics

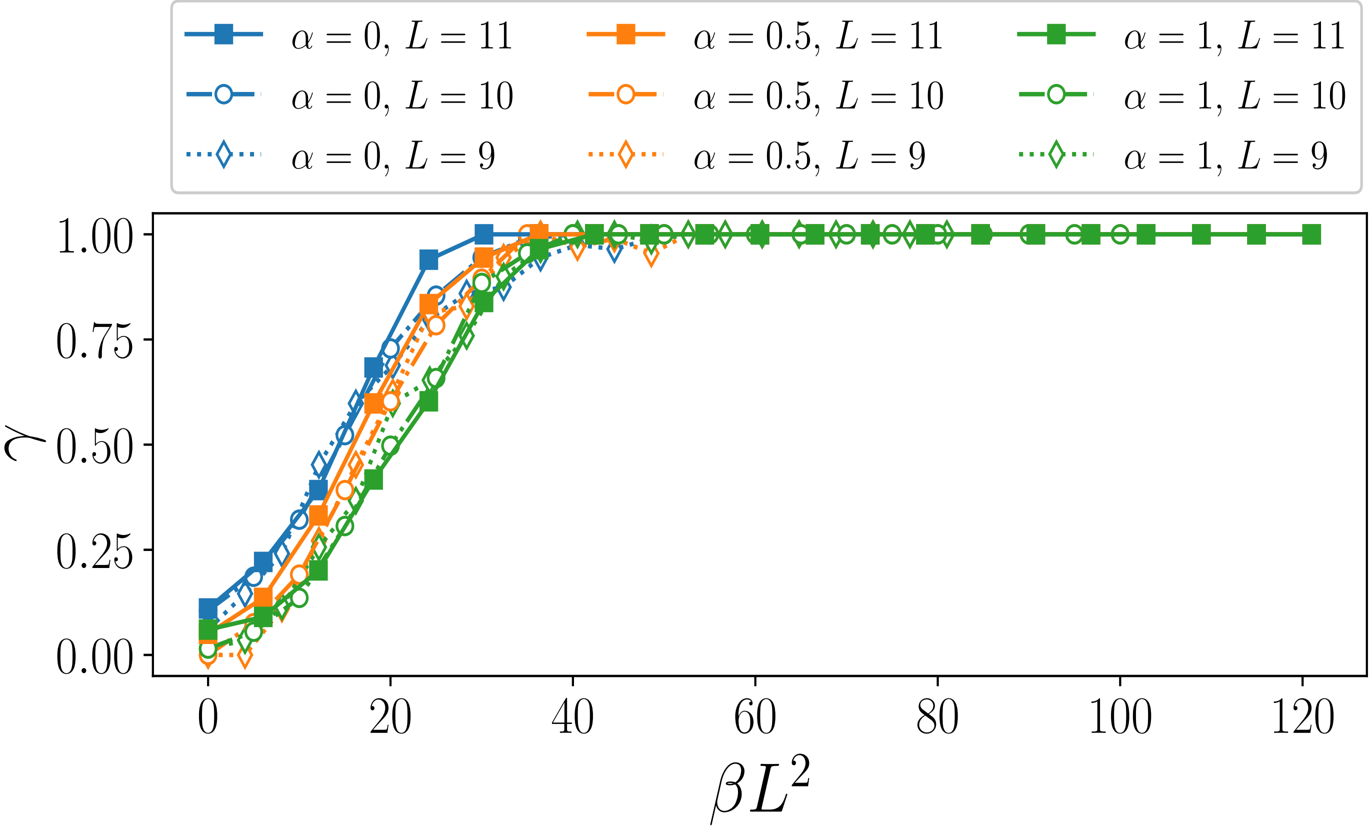


Transition to chaos



Dimension L=11 ~ 28000, total eigenvalues per point ~ 700000

Transition to chaos



Dimension L=11 ~ 28000, total eigenvalues per point ~ 700000

Conclusions

- New family of integrable spin chain quantum Liouvillians.
- Simple transition Liouvillian.
- Characterization of the transition to chaos.

  - Simple fit to interpolating formula with fractional level repulsion.

  - Spacing ratios.

arXiv:2102.13452