PHASE-SPACE LOCALIZATION MEASURES AND QUANTUM SCARS

Dicke-Team Annual Meeting:

Chaos and localization in quantum many-body systems

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Outline

Introduction

Localization Measures of Quantum States

In physics, it is used the exponential of a given entropy to quantify the spreading degree of a given state

$$\text{Spreading degree} = \exp(\text{Entropy}).$$

The general connection comes from the Boltzmann entropy

$$\text{Entropy} \propto \ln(\Omega) \Longleftrightarrow \exp(\text{Entropy}) \propto \Omega,$$

where Ω defines the number of accessible microstates.

A localization (delocalization) measure quantifies the concentration (spreading) degree of a quantum state in a discrete or continuous basis.

Localization Measures in Discrete Bases

A usual localization measure in a Hilbert space $\mathcal{H}$ is the participation ratio

$$P_R = \left(\sum_{k=1}^{D} p_k^2 \right)^{-1} \in [1, D], \qquad (1)$$

where $p_k = |\langle \phi_k | \Psi \rangle|^2$ (with $\sum_k p_k = 1$) defines the probability to find the arbitrary state $|\Psi\rangle$ in each state $|\phi_k\rangle$ of a given discrete basis $\mathcal{B}$ of dimension D.

Localization Measures in Continuous Bases

The participation ratio can be generalized to a continuous basis. Considering a basis $\mathcal{B}(k)$ with parameter $k \in \mathcal{D}$ defined in the continuous space $\mathcal{D}$, it is possible to define a participation ratio

$$P_R = \left(\int_{\mathcal{D}} dk \, p^2(k) \right)^{-1} \in \left(0, \mathcal{V}_{\mathcal{D}} = \int_{\mathcal{D}} dk \right], \qquad (2)$$

where $p(k) = |\langle k|\Psi\rangle|^2$ (with $\int_{\mathcal{D}} dk \, p(k) = 1$) is the probability distribution to find the arbitrary state $|\Psi\rangle$ in each point k of the continuous space $\mathcal{D}$.

Entropy Associated to the Participation Ratio

The Rényi entropy S_d of order d ($d \geq 0$, $d \neq 1$), defines the localization measure

$$\mathcal{L}_d = e^{S_d} = \left(\sum_{k=1}^{D} p_k^d \right)^{1/(1-d)} \quad \text{with } S_d = \frac{1}{1-d} \ln \left(\sum_{k=1}^{D} p_k^d \right), \qquad (3)$$

whose particular case $d = 2$ is $P_R = \mathcal{L}_2 = e^{S_2} = \left(\sum_{k=1}^{D} p_k^2 \right)^{-1}$.

The limiting case $d \to 1$ is associated with the Shannon entropy $S_1 = \lim_{d\to1}(S_d)$, and defines the localization measure

$$\mathcal{L}_1 = e^{S_1} = exp \left(- \sum_{k=1}^{D} p_k \ln(p_k) \right) \text{ with } S_1 = - \sum_{k=1}^{D} p_k \ln(p_k). \qquad (4)$$

Localization Measures in Discrete and Continuous Bases

	Basis $\mathcal{B}$ with dimension D	Basis $\mathcal{B}(k)$ with $k \in \mathcal{D}$
$\mathcal{L}_d = e^{S_d}$	$\left(\sum_{k=1}^{D} p_k^d\right)^{1/(1-d)}$	$\left(\int_{\mathcal{D}} dk\, p^d(k)\right)^{1/(1-d)}$
$\mathcal{L}_1 = e^{S_1}$	$exp\left(-\sum_{k=1}^{D} p_k \ln(p_k)\right)$	$exp\left(-\int_{\mathcal{D}} dk\, p(k) \ln[p(k)]\right)$
$P_R = e^{S_2}$	$\left(\sum_{k=1}^{D} p_k^2\right)^{-1}$	$\left(\int_{\mathcal{D}} dk\, p^2(k)\right)^{-1}$
Rényi of order d: S_d $(d \geq 0,\ d \neq 1)$, Shannon: S_1 $(d \to 1)$		

Unbounded Continuous Spaces

When the continuous space $\mathcal{D}$ is unbounded, then its volume is infinite

$$\mathcal{V}_{\mathcal{D}} = \int_{\mathcal{D}} dk = \infty, \tag{5}$$

and the distribution $p(k)$ is arbitrarily delocalized.

Choosing bounded subspaces $\mathcal{S} \subset \mathcal{D}$ with finite volume $\mathcal{V}_{\mathcal{S}} < \mathcal{V}_{\mathcal{D}}$ it is possible to define relative localization measures

$$\mathcal{L}_{\mathcal{D}} \to \frac{\mathcal{L}_{\mathcal{S}}}{\mathcal{V}_{\mathcal{S}}} \in (0, 1]. \tag{6}$$

Spin-Boson Dicke Model

The Dicke model is given by the Hamiltonian

$$\hat{H}_D = \omega \hat{a}^\dagger \hat{a} + \omega_0 \hat{J}_z + \frac{2\gamma}{\sqrt{\mathcal{N}}} \hat{J}_x (\hat{a}^\dagger + \hat{a}), \tag{7}$$

where:

- $\hat{a}$ ($\hat{a}^\dagger$) is the annihilation (creation) operator.
- $\hat{J}_{x,y,z} = \sum_{n=1}^{\mathcal{N}} \hat{\sigma}^n_{x,y,z}/2$ are the collective pseudo-spin operators, and $\hat{\sigma}^n_{x,y,z}$ are the Pauli matrices.
- $\mathcal{N}$ is the total number of atoms within the system.
- ω is the radiation frequency of the electromagnetic field.
- ω_0 is the transition frequency between two atomic levels.
- γ is the coupling strength, whose critical value $\gamma_c = \sqrt{\omega\omega_0}/2$ separates two phases: normal ($\gamma < \gamma_c$) and super-radiant ($\gamma > \gamma_c$).

Classical Limit of the Spin-Boson Dicke Model

Taking the expectation value of $\hat{H}_D$ under Glauber $|q,p\rangle = \hat{D}(q,p)|0\rangle$ and Bloch $|Q,P\rangle = \hat{R}(Q,P)|j,-j\rangle$ coherent states, a classical Hamiltonian can be obtained

$$\begin{aligned} h_{\text{cl}}(\mathbf{x}) =& \frac{\langle \mathbf{x}|\hat{H}_D|\mathbf{x}\rangle}{j} = \frac{\langle q,p| \otimes \langle Q,P|\hat{H}_D|q,p\rangle \otimes |Q,P\rangle}{j} \\ =& \frac{\omega}{2}\left(q^2+p^2\right) + \frac{\omega_0}{2}\left(Q^2+P^2\right) - \omega_0 + 2\gamma q Q\sqrt{1-\frac{Q^2+P^2}{4}}, \end{aligned} \tag{8}$$

with scaled energy $\epsilon = E/j$. The scaling $j = \mathcal{N}/2$ defines a 4-dimensional phase space in the canonical variables $\mathbf{x} = (q,p;Q,P)$, which is independent of the system size.

Localization Measures in the Dicke-Model Phase Space

We call these localization measures the *"Rényi occupations of order α"* of the quantum state $\hat{\rho}$, which are given by

$$\mathfrak{L}_\alpha(\mathcal{S},\hat{\rho}) = \frac{1}{\mathcal{V}_\mathcal{S}}\left(\int_\mathcal{S} d\mathcal{V}_\mathcal{S}(\mathbf{x})\Phi^\alpha_{\hat{\rho}}(\mathbf{x})\right)^{1/(1-\alpha)}, \tag{9}$$

$$\mathfrak{L}_1(\mathcal{S},\hat{\rho}) = \frac{1}{\mathcal{V}_\mathcal{S}}\exp\left(-\int_\mathcal{S} d\mathcal{V}_\mathcal{S}(\mathbf{x})\Phi_{\hat{\rho}}(\mathbf{x})\ln[\Phi_{\hat{\rho}}(\mathbf{x})]\right), \tag{10}$$

where $\mathcal{S}$ denotes the bounded subspace with finite volume $\mathcal{V}_\mathcal{S}$, and $\Phi_{\hat{\rho}}(\mathbf{x})$ is the probability distribution defined in the coherent-state (overcomplete) basis $|\mathbf{x}\rangle = |q,p\rangle \otimes |Q,P\rangle$.

Rényi Occupations of Order α over Classical Energy Shells

	Classical energy shells ϵ
Subspace	$\mathcal{M}_\epsilon = \{\mathbf{x} \mid h_{\text{cl}}(\mathbf{x}) = \epsilon\}$
Volume	$\mathcal{V}_{\mathcal{M}_\epsilon} = \int_{\mathcal{M}_\epsilon} \delta(h_{\text{cl}}(\mathbf{x}) - \epsilon) d\mathbf{x}$
Distribution	$\Phi_{\hat{\rho}}(\mathbf{x}) = \frac{1}{C_{\mathcal{M}_\epsilon}} \mathcal{Q}_{\hat{\rho}}(\mathbf{x})$ $\mathcal{Q}_{\hat{\rho}}(\mathbf{x}) = \langle \mathbf{x} \vert \hat{\rho} \vert \mathbf{x} \rangle$ $C_{\mathcal{M}_\epsilon} = \int_{\mathcal{M}_\epsilon} d\mathcal{V}_{\mathcal{M}_\epsilon}(\mathbf{x}) \mathcal{Q}_{\hat{\rho}}(\mathbf{x})$

- D. Villaseñor, S. Pilatowsky-Cameo, M. A. Bastarrachea-Magnani, S. Lerma-Hernández, and J. G. Hirsch, Phys. Rev. E **103**, 052214 (2021).

Maximally Delocalized States

A quantum state $\hat{\rho}$ is called maximally delocalized in the classical energy shell ϵ when its Rényi occupation is

$$\mathfrak{L}_\alpha(\mathcal{M}_\epsilon, \hat{\rho}) \approx \mathfrak{L}_\alpha^{\mathrm{max}} \equiv \Gamma(1+\alpha)^{1/(1-\alpha)}, \tag{11}$$

where Γ is the Gamma function.

This value is the averaged level of localization attained by random states of the form $\hat{\rho}_{\mathrm{R}} = |\psi_{\mathrm{R}}\rangle\langle\psi_{\mathrm{R}}|$, where $|\psi_{\mathrm{R}}\rangle = \sum_k c_k |E_k\rangle$ and $\hat{H}_D|E_k\rangle = E_k|E_k\rangle$, and it is the maximum level of delocalization attained by a pure state.

- S. Pilatowsky-Cameo, D. Villaseñor, M. A. Bastarrachea-Magnani, S. Lerma-Hernández, L. F. Santos, and J. G. Hirsch, arXiv:2107.06894 (2021).

Measure of Localization Λ_α

The maximum value of the Rényi occupation $\mathfrak{L}_\alpha^{\text{max}}$ allows us to define the following measure

$$\Lambda_\alpha(\mathcal{M}_\epsilon, \hat{\rho}) = \frac{\mathfrak{L}_\alpha^{\text{max}}}{\mathfrak{L}_\alpha(\mathcal{M}_\epsilon, \hat{\rho})} \in [1, \infty), \tag{12}$$

where a value $\Lambda_\alpha = 1$ defines a state as delocalizaded as a random state, while a finite value $\Lambda_\alpha = L$ defines a state L times more localized than a random state.

- S. Pilatowsky-Cameo, D. Villaseñor, M. A. Bastarrachea-Magnani, S. Lerma-Hernández, L. F. Santos, and J. G. Hirsch, arXiv:2107.06894 (2021).

Localization of Eigenstates

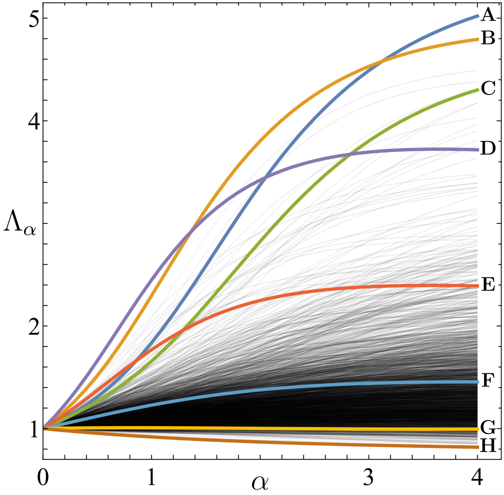
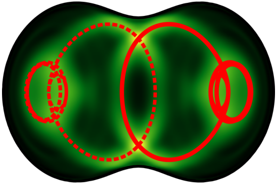


Figure: $\Lambda_\alpha(\epsilon_k, \hat{\rho}_k)$ for 2437 eigenstates $\hat{\rho}_k = |E_k\rangle\langle E_k|$ located in the chaotic energy region $\epsilon_k = E_k/j \in [-0.6, -0.4]$. System size: $j = 100$.

Quantum Scars and Unstable Periodic Orbits

The term "scar" was coined to define the imprints left by the classical domain (unstable periodic orbits), which are captured by the quantum domain (quasiprobability distributions: Wigner, Husimi).



- Steven William McDonald, *Wave Dynamics of Regular and Chaotic Rays*, (Dissertation, Ph. D. in Physics, University of California, Berkeley, 1983).
- Eric J. Heller, Phys. Rev. Lett. **53**, 1515 (1984).

α-Moments of the Husimi Function

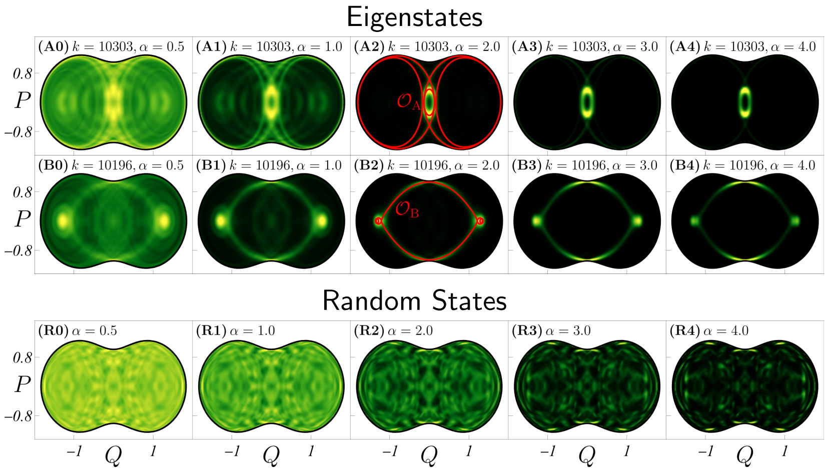


Figure: Husimi projection of eigenstates $\hat{\rho}_k = |E_k\rangle\langle E_k|$ and random states $\hat{\rho}_R = |\psi_R\rangle\langle\psi_R|$ for different orders $\alpha = 0.5, 1, 2, 3, 4$. The red curves identify the unstable periodic orbits. System size: $j = 100$.

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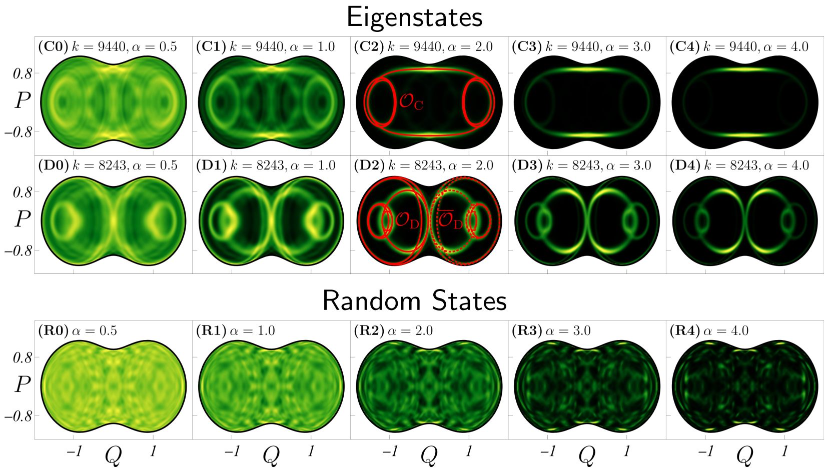

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Classical Dynamics and Unstable Periodic Orbits

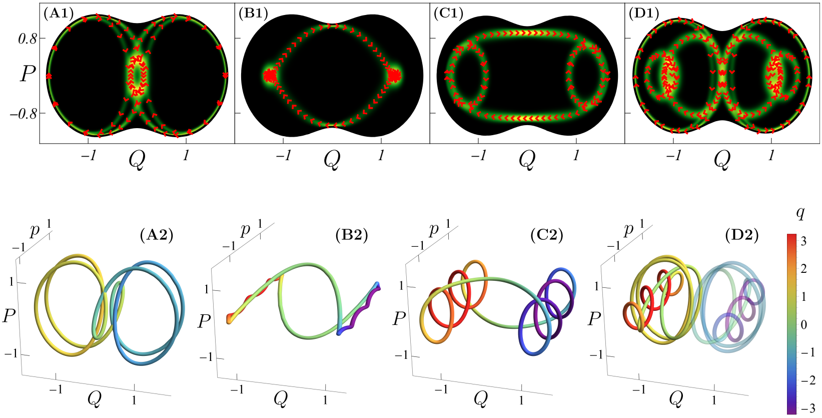

Figure: **Top panels:** Husimi projection of the tubular state $\hat{\rho}_{\mathcal{O}} = \frac{1}{T}\int_0^T dt|\mathbf{x}(t)\rangle\langle\mathbf{x}(t)|$, where $\mathcal{O} = \{\mathbf{x}(t)|t \in [0, T)\}$ represents the unstable periodic orbit of period T. System size: $j = 100$. **Bottom panels:** 3D plots of the unstable periodic orbits.

Conclusions

- The analysis of the α-moments of the Husimi function is a useful tool to identify visually unstable periodic orbits.
- The study of the classical dynamics of the unstable periodic orbits helps in the understanding of the localization of states, since in regions where the dynamics is slower the values of the Husimi function are larger.
- Some eigenstates of the model are significantlly more localized than most. This high level of localization is caused by quantum scars.
- The studies developed to identify unstable periodic orbits in the spin-boson Dicke model can be extended to other systems with an unbounded phase space.