Multifractality and lack of self-averaging around the many-body localization transition

E. Jonathan Torres-Herrera

Institute of Physics

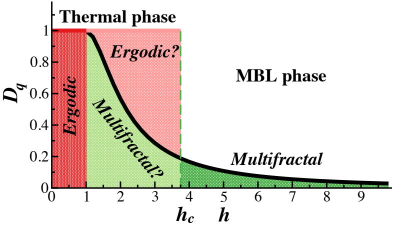
Benemérita Universidad Autónoma de Puebla

Puebla, México

Outline

We show that multifractal states could exist in a region before the many-body localization transition.

- Multifractal states at the metal-insulator (Anderson) transition.
- Localization in interacting systems: Many-body localization (MBL).
- Self-averaging around the MBL transition.
- Multifractality: From the 1D Fibonacci lattice to many-body systems.



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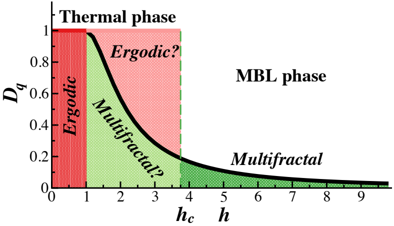
Lea F. Santos
Yeshiva University



Andrei Solórzano
Tecnológico de Monterrey

A. Solórzano, L. F. Santos, **EJTH**, *arXiv* (2020).

Metal-insulator (Anderson) transition

The Anderson model is written as

$$H = \sum_i \varepsilon_i |i\rangle \langle i| + \sum_{\langle ij \rangle} J |i\rangle \langle j|$$

ε_i **are independent and uniform random numbers in** $(-\frac{W}{2}, \frac{W}{2})$

- In 1D all states are exponentially localized for any disorder strength W.
- In 2D and 3D there is transition from a phase with extended states to a phase with localized states.

P. W. Anderson, Phys. Rev. **109**, 1492 (1958).

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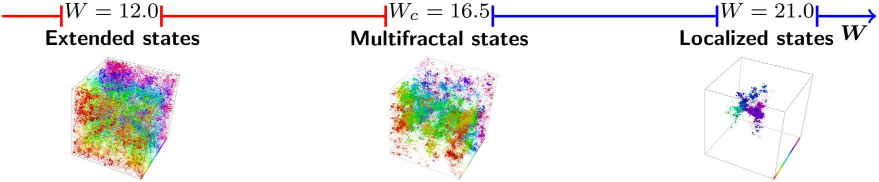

P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
Images from O. Schenk, M. Bollhöfer, R. A. Römer, SIAM Rev. **50**, 91 (2008).

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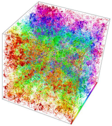
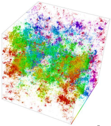
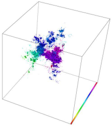
$W = 12.0$ $\quad$ $W_c = 16.5$ $\quad$ $W = 21.0$ $\quad$ W

Extended states $\quad$ **Multifractal states** $\quad$ **Localized states**







Generalized inverse participation ratios

$$\text{IPR}_q = \sum_{n=1}^{\mathcal{N}} |\langle \phi_n|\psi_\alpha\rangle|^{2q} \propto \mathcal{N}^{-D_q(q-1)}$$

P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
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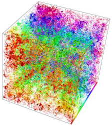	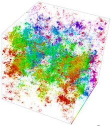	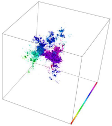

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$D_q = 1$	$0 < D_q < 1$	$D_q = 0$

P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
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Many-body localization (MBL)

The standard model to study MBL

$$\boldsymbol{H = \sum_{k=1}^{L} \left(S_k^x S_{k+1}^x + S_k^y S_{k+1}^y + S_k^z S_{k+1}^z \right) + \sum_{k=1}^{L} h_k S_k^z}$$

- $S_k^{x,y,z} = \frac{1}{2}\sigma_k^{x,y,z}$ are spin-1/2 operators.
- h_k are random numbers from an uniform distribution in $(-h, h)$.
- The Hamiltonian conserves total spin in z-direction, $\mathcal{S}^z = \sum_k S_k^z$.
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Level spacing distribution

$$s = (E_{\alpha+1} - E_\alpha)/\Delta$$

Poisson

$$P_{\mathsf{P}}(s) = e^{-s}$$

Wigner-Dyson

$$P_{\mathsf{WD}}^{\mathsf{GOE}} = \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right)$$

We quantify the degree of level repulsion with

$$\eta = \frac{\int_0^{s_0} [P(s) - P_{\mathsf{WD}}^{\mathsf{GOE}}(s)] ds}{\int_0^{s_0} [P_{\mathsf{P}}(s) - P_{\mathsf{WD}}^{\mathsf{GOE}}(s)] ds} = \begin{cases} 1, & \text{Poisson.} \\ 0, & \text{GOE.} \end{cases}$$

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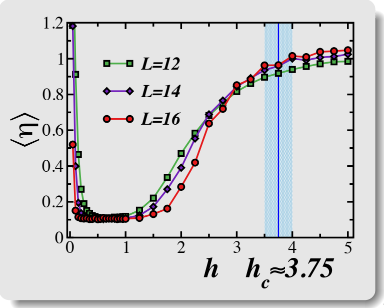

Multifractality at (around) the MBL phase

A single critical point or an extended phase where the eigenstates are multifractal?

Ergodicity breaking in a model showing many-body localization

A. De Luca[1,2] and A. Scardicchio[3,4]

Published 15 February 2013 • Copyright © EPLA, 2013

EPL (Europhysics Letters), Volume 101, Number 3

Universal Behavior beyond Multifractality in Quantum Many-Body Systems

David J. Luitz, Fabien Alet, and Nicolas Laflorencie

Phys. Rev. Lett. **112**, 057203 – Published 6 February 2014

Dynamics at the many-body localization transition

E. J. Torres-Herrera and Lea F. Santos

Phys. Rev. B **92**, 014208 – Published 24 July 2015

Many-body localization edge in the random-field Heisenberg chain

David J. Luitz, Nicolas Laflorencie, and Fabien Alet

Phys. Rev. B **91**, 081103(R) – Published 9 February 2015

Theory of the Many-Body Localization Transition in One-Dimensional Systems

Ronen Vosk, David A. Huse, and Ehud Altman

Phys. Rev. X **5**, 031032 – Published 14 September 2015

Many-body localization phase transition: A simplified strong-randomness approximate renormalization group

Liangsheng Zhang, Bo Zhao, Trithep Devakul, and David A. Huse

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Spectral statistics across the many-body localization transition

Maksym Serbyn and Joel E. Moore

Phys. Rev. B **93**, 041424(R) – Published 29 January 2016

Thouless energy and multifractality across the many-body localization transition

Maksym Serbyn, Z. Papić, and Dmitry A. Abanin

Phys. Rev. B **96**, 104201 – Published 6 September 2017

Extended nonergodic states in disordered many-body quantum systems

E. J. Torres-Herrera, Lea F. Santos

First published: 13 January 2017 | https://doi.org/10.1002/andp.201600284 | Citations:

Renormalization-group study of the many-body localization transition in one dimension

Alan Morningstar and David A. Huse

Phys. Rev. B **99**, 224205 – Published 21 June 2019

Kosterlitz-Thouless scaling at many-body localization phase transitions

Philipp T. Dumitrescu, Anna Goremykina, Siddharth A. Parameswaran, Maksym Serbyn, and Romain Vasseur

Phys. Rev. B **99**, 094205 – Published 22 March 2019

Many-body localization near the critical point

Alan Morningstar, David A. Huse, and John Z. Imbrie

Phys. Rev. B **102**, 125134 – Published 21 September 2020

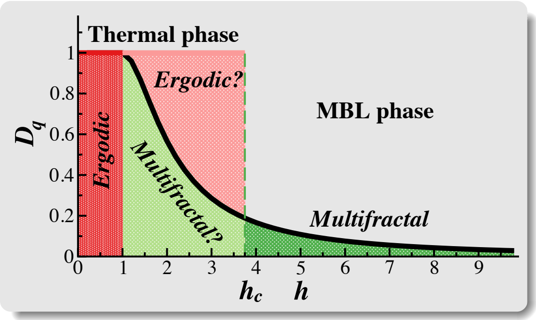
Multifractal Scalings Across the Many-Body Localization

Nicolas Macé, Fabien Alet, and Nicolas Laflorencie

Phys. Rev. Lett. **123**, 180601 – Published 29 October 2019

Multifractality at (around) the MBL phase

A single critical point or an extended phase where the eigenstates are multifractal?

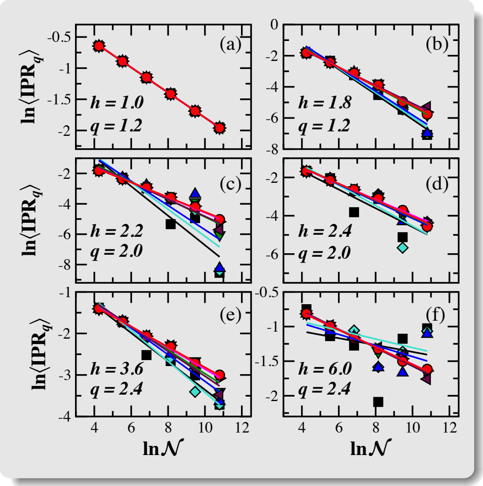


Finite size effects **Finite times**

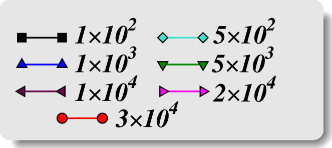
We address self-averaging

Generalized inverse participation ratios

$$\ln \langle \text{IPR}_q \rangle \ \text{ vs } \ -D_q(q-1)\ln \mathcal{N}$$



- We average, $\langle \cdots \rangle$, over states and disorder realizations.



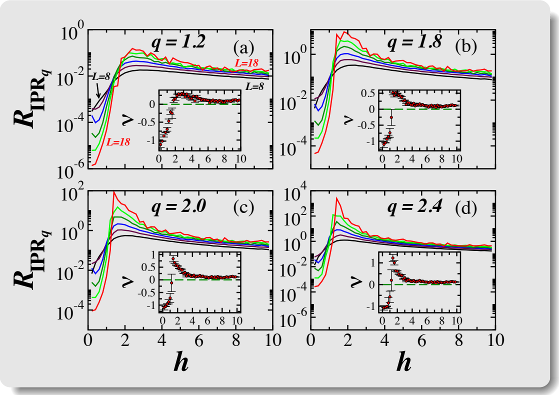
- Small ensembles lead to larger fluctuations.
- $h \leq 1$, independence on number of samples.
- $h > 1$, fluctuations increase with system size.

Lack of self-averaging?

Self-averaging

Operational definition

$$\mathcal{R}_{\mathcal{O}} = \frac{\langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2}{\langle \mathcal{O} \rangle^2} \propto \mathcal{N}^{\nu}, \quad \begin{cases} \nu < 0, & \text{self-averaging.} \\ \nu \geq 0, & \text{lack of self-averaging.} \end{cases}$$



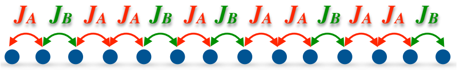
Strong lack of self-averaging around $h_c \approx 3.75$

Generalized dimensions: 1D Fibonacci lattice

Fibonacci sequence

1 A $\rightarrow$ B
2 B $\rightarrow$ AB
3 AB
4 BAB
5 ABBAB
6 BABABBAB
7 ABBABBABABBAB

$$H = \sum_i \varepsilon \, |i\rangle \langle i| + \sum_i J_i \, |i\rangle \langle i+1|$$



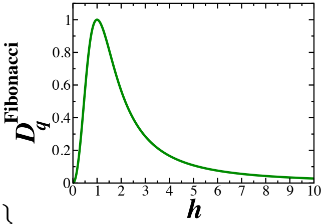
For the state with energy zero

$$D_q^{\text{Fibonacci}} = \frac{1}{3(q-1)\ln\sigma} \left\{ q\ln[\lambda(h^2)] - \ln[\lambda(h^{2q})] \right\}$$

$$h = J_B / J_A$$

and

$$\sigma = (\sqrt{5}+1)/2, \;\; \lambda(h) = \frac{1}{2h} \left\{ (h+1)^2 + \left[(h+1)^4 + 4h^2 \right]^{1/2} \right\}$$

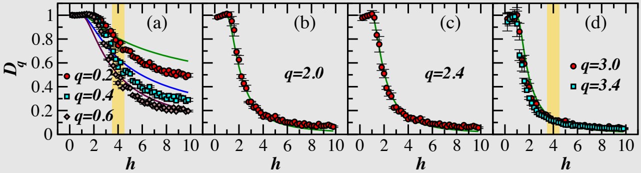


Could we use $D_q^{\text{Fibonacci}}$ for many-body systems?

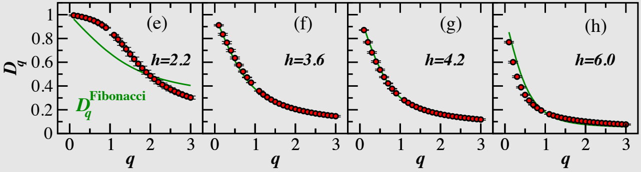
T. Fujiwara, M. Kohmoto, and T. Tokihiro, Phys. Rev. B **40**, 7413 (1989).

$D_q^{\text{Fibonacci}}$ vs D_q^{spin}

$$\Theta(h-1)D_q^{\text{Fibonacci}} + \Theta(h-1)$$



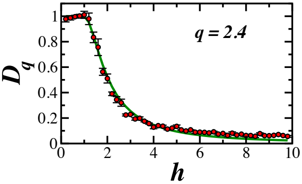
Very good agreement close to $h_c \approx 3.75$

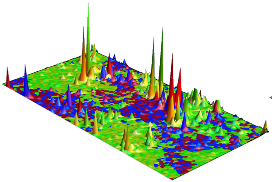


Multifractality holds when D_q is a nonlinear function of q

Summary

- Besides finite size effects, self-averaging is an important property to consider in theoretical and experimental studies.
- Our results are in line with the existence of an intermediate phase with multifractal states.
- D_q for the disordered spin-$1/2$ agrees very well with $D_q^{\text{Fibonacci}}$.
- We observed a strong correlation between multifractality and lack of self-averaging.





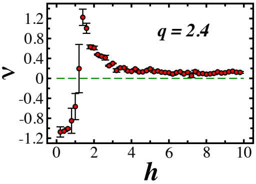


Image by Yang-Zhi Chou and Matthew Foster/Rice University.

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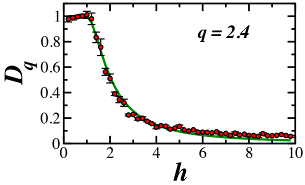

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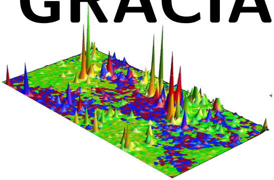

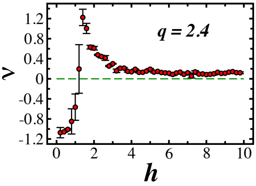

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