Localization measures in quantum chaotic billiards

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Other members:

- Dr. Qian Wang (talk tomorrow)

Outline

Classical billiards[1]

- Free moving particle on billiard table $\mathcal{B} \subset \mathbb{R}^2$.
- Specular reflection with boundary: $\alpha = \alpha'$.
- Energy conserved

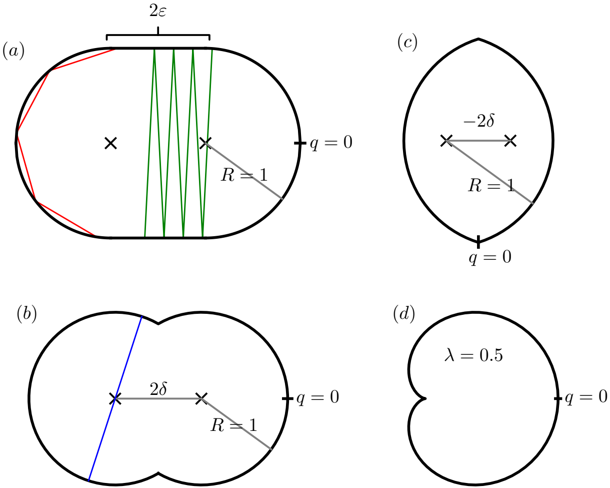
$$H = \frac{\boldsymbol{g}^2}{2m} + V(\boldsymbol{r}). \tag{1}$$

- Potential

$$V(\boldsymbol{r}) = \begin{cases} 0 & \boldsymbol{r} \in \mathcal{B}, \\ \infty & \boldsymbol{r} \notin \mathcal{B}. \end{cases} \tag{2}$$

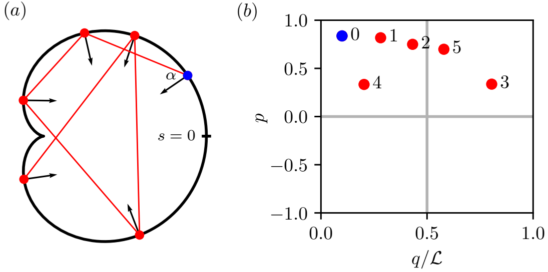
- Trajectory is a series of collisions linked by straight lines.

Billiard tables



Poincaré-Birkhoff coordinates

- The dynamics may be described as a discreet mapping.
- Poincaré-Birkhoff coordinates q, $p = \sin(\alpha)$.
- The phase space is a cylinder $\mathcal{M} = [0, \mathcal{L}] \times (-1, 1)$



Quantum billiards[2]

- Quantum particle trapped inside billiard table.
- Hamiltonian (units $\hbar^2/2m = 1$)

$$H = -\nabla^2 + V(\boldsymbol{r}), \tag{3}$$

- Stationary problem - Helmholtz equation, $E = k^2$

$$\left(\nabla^2 + k^2\right)\psi_k(\boldsymbol{r}) = 0, \tag{4}$$

- Boundary condition $\psi|_{\partial\mathcal{B}} = 0$.
- Problem may be converted to a boundary integral equation.

Boundary functions

- Boundary functions

$$u\left(q\right)=\boldsymbol{n}\cdot\nabla_{\boldsymbol{r}}\psi\left(\boldsymbol{r}\left(q\right)\right). \tag{5}$$

- Integral equation on the boundary

$$u\left(q\right)=-2\oint \mathrm{d}l u\left(l\right)\boldsymbol{n}\cdot\nabla_{\boldsymbol{r}}G\left(\boldsymbol{r},\boldsymbol{r}\left(l\right)\right). \tag{6}$$

- Wavefunction may be reconstructed from the boundary function

$$\psi_k\left(\boldsymbol{r}\right)=-\oint \mathrm{d}l u_k\left(l\right)G\left(\boldsymbol{r},\boldsymbol{r}\left(l\right)\right). \tag{7}$$

- Scaling enables efficient numerical solution.[3,4]

Poincaré-Husimi functions[5]

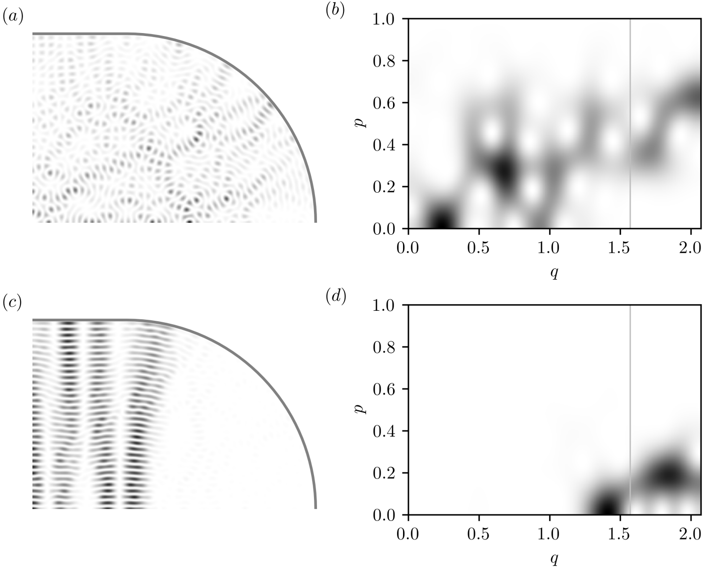
- We represent eigenstates as probability distributions in the phase space.
- Periodic coherent states at PB points (q,p)

$$c_{(q,p),k}(l) = \sum_{m\in\mathbb{Z}} \exp\left[ikp\left(l-q+m\mathcal{L}\right)\right]\exp\left[-\frac{k}{2}\left(l-q+m\mathcal{L}\right)^2\right]. \tag{8}$$

- Poincaré-Husimi functions

$$H_n(q,p) = \frac{1}{Q_n}\left|\oint_{\partial B} c_{(q,p),k_n}(l)\, u_n(l)\, \mathrm{d}l\right|^2. \tag{9}$$

Example - Stadium billiard



Chaotic eigenstates

- Wigner and Husimi representations - quantum phase space.
- Principle of uniform semiclassical condensation: eigenstates ”live” on classical sub-components.
- Eigenstates may be separated into regular and chaotic states.
- Quantum ergodicity - chaotic eigenstates uniformly cover the chaotic component.
- Dynamical localization - chaotic eigenstates cover only small part of the chaotic component.

Quantum dynamical localization

- Localization: wavefunction is large on only part of the available space.
- Dynamical localization:[6,7] quantum interference suppresses diffusion of wavepackets.
- Discrete spectrum is resolved after Heisenberg time $t_H = 2\pi\hbar/\Delta E$.
- Quantum dynamics follows classical dynamics until t_H.
- Classical transport time t_T, typical time scale of classical diffusion.
- We expect eigenstates are localized if $t_T > t_H$.
- Control parameter

$$\alpha = \frac{t_H}{t_T} = \frac{\mathcal{L}k}{\pi N_T}. \tag{10}$$

Localization measures

- Information entropy

$$S_n = -\int_0^{\mathcal{L}} \mathrm{d}q \int_{-1}^{1} \mathrm{d}p H_n(q,p) \ln\left(H_n(q,p)\right). \tag{11}$$

- Entropy localization measure

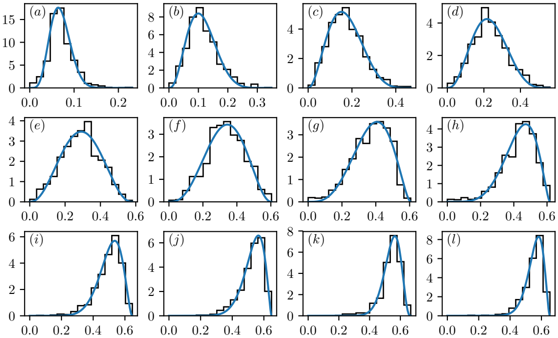
$$A_n = \frac{\exp\left(S_n\right)}{N_c}. \tag{12}$$

- N_c number of grid points in the chaotic component.
- $0 < A < A_0 < 1$ small for localized states, large for extended states.
- What is the distribution of A of the chaotic eigenstates?
- How does A depend on α?

Distributions of A - Stadium billiards

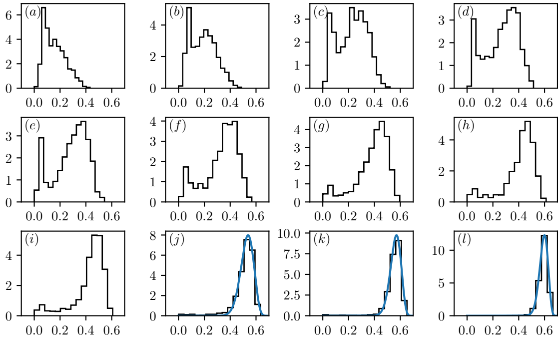
- Beta distribution

$$P(A) = CA^a(A_0 - A)^b. \tag{13}$$



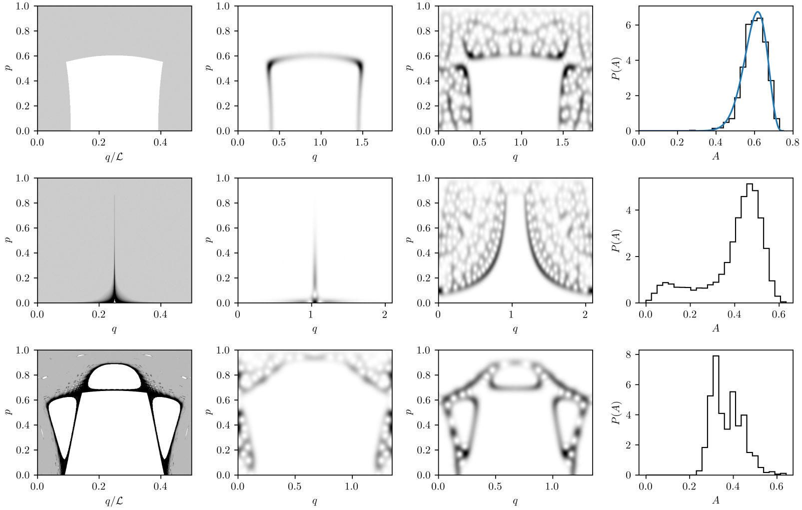
- Found in systems without stickiness.

Distributions of A - Robnik billiards



- Non-universal distributions for chaotic states in mixed-type billiards.

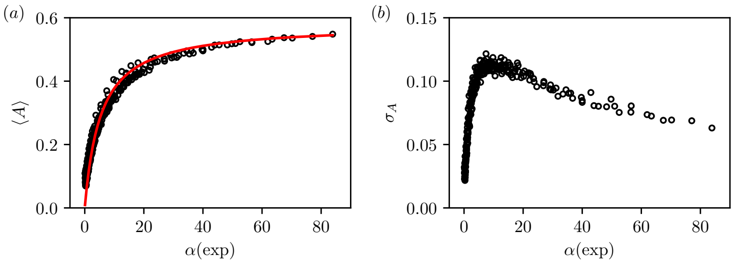
Quantum stickiness - Lemon billiards



Moments of A - stadium billiards

- Empirically rational function

$$\langle A \rangle = A_\infty \frac{\zeta\alpha}{1+\zeta\alpha}. \tag{14}$$



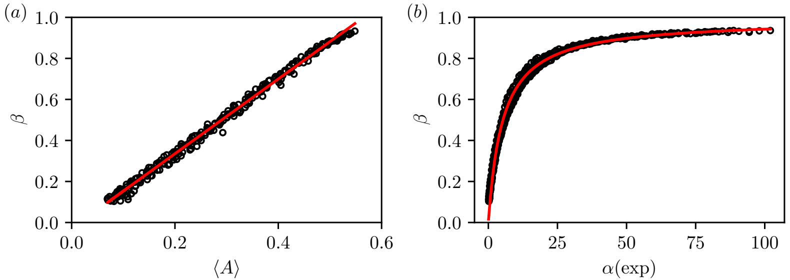
Localization and spectral statistics

- Localization diminishes level repulsion.
- Empirical Brody distribution

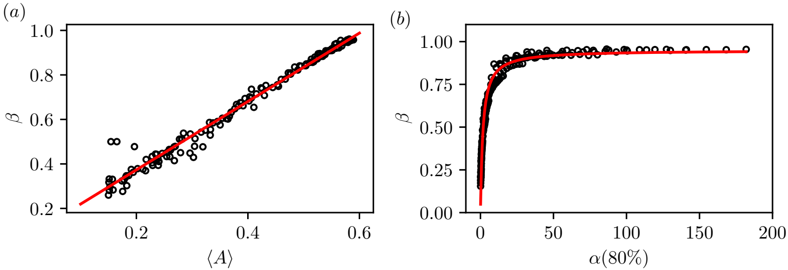
$$P_B(s) = c(\beta+1)\, s^{\beta} \mathrm{e}^{-cs^{\beta+1}} \tag{15}$$

- In mixed-type systems Berry-Robnik-Brody distribution.
- Additional parameter, classical Liouville measures ρ_c, $\rho_r = 1 - \rho_c$.
- How does β depend on α?
- How does β depend on A?

β dependence - stadium billiards



β dependence - Robnik billiards



Other localization measures

- PH function $H_{ij}^n = H_n(q_i, p_j)$
- Normalized inverse participation ratio

$$R_n = \frac{1}{N_c} \frac{1}{\sum_{ij} (H_{ij}^n)^2}. \tag{16}$$

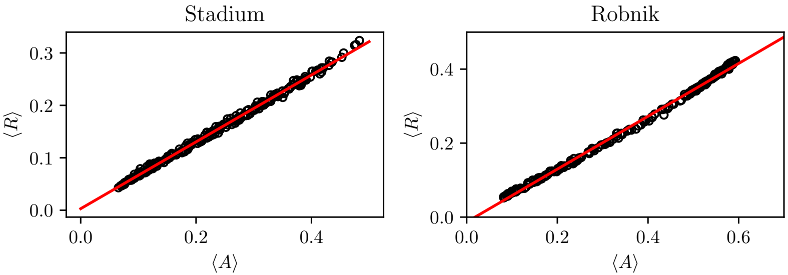
- Correlation localization measure

$$C_{nm} = \frac{1}{Q_n Q_m} \sum_{ij} H_{ij}^n H_{ij}^m. \tag{17}$$

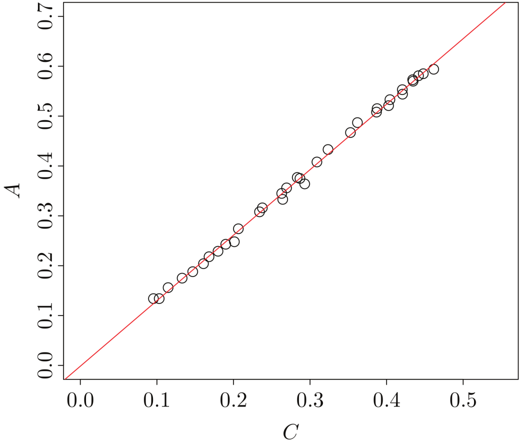
- Normalisation $Q_n = \sqrt{\sum_{ij} (H_{ij}^n)^2}$

R vs A

- Linear dependence of correlation measures



A vs C - Robnik billiard[8]



Summary and Conclusion

- Numerical study of chaotic billiard eigenstates.[9]
- The parameter $\alpha = \frac{t_H}{t_T}$ controls the onset of localization.
- Localization measures defined via PH functions.
- Beta distributions of entropy localization measures A.
- Classical stickiness alters A distributions.
- Level spacing distributions well described by Brody or Berry-Robnik-Brody.
- Mean A and β rational function of α.
- Linear dependence of averaged localization measures A, R and C.
- Python Quantum Billiard library available on GitHub.[10]

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10 v. Lozej and B. Batistić, "Quantum billiards." Available at https://github.com/clozej/quantum-billiards/tree/crt_public.