The k-body bosonic Embedded Gaussian Ensemble: Ergodicity in the dense limit

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L. Benet

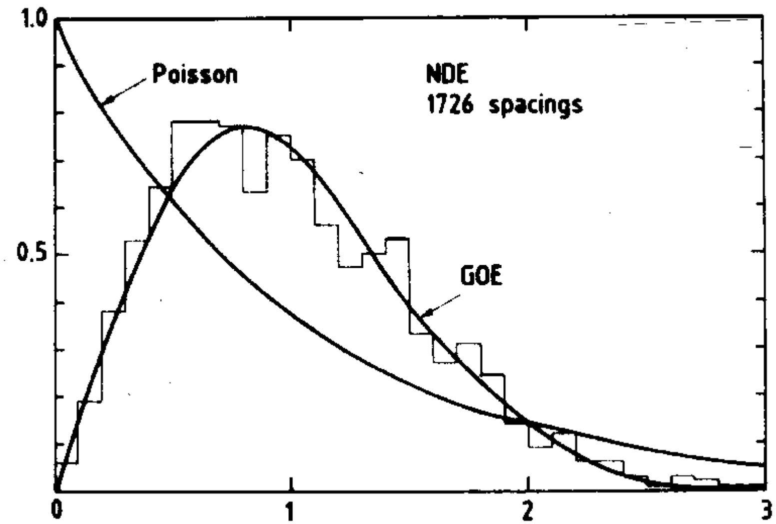
Motivation: Random Matrix Theory

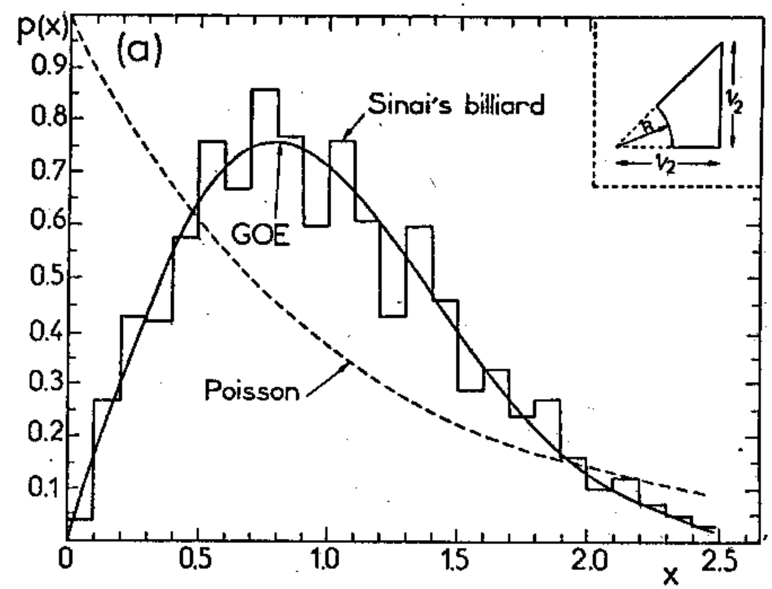
EP Wigner, Ann. Math **53**, 36 (1951)

- RMT was introduced to describe the statistical properties of the compound nucleus: spectral statistics
- The Hamiltonian of the system is replaced by an ensemble of random Hamiltonians with the same symmetry properties: GOE, GUE, GSE
- For $N \to \infty$ RMT makes precise parameter-free predictions: semicircle law, level repulsion, long-range stiffness
- Ergodicity and stationarity

Motivation: Random Matrix Theory

RMT is successful in describing the statistical properties of the spectrum of many–body systems and of quantized classically chaotic two d.o.f. systems





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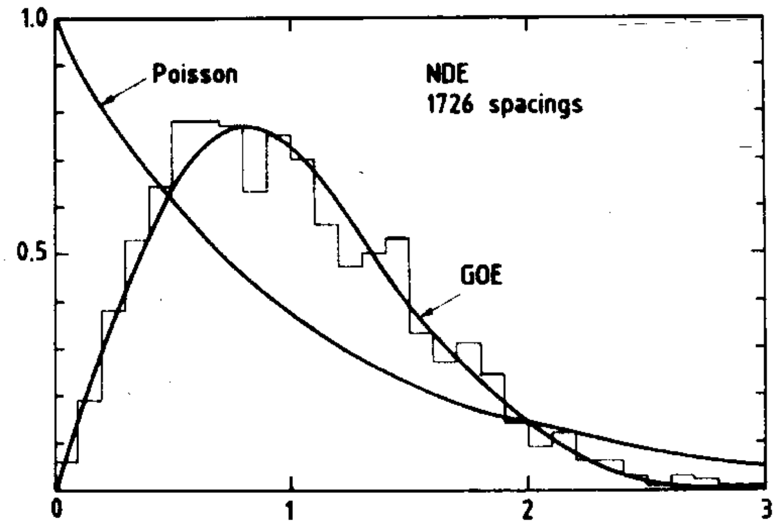

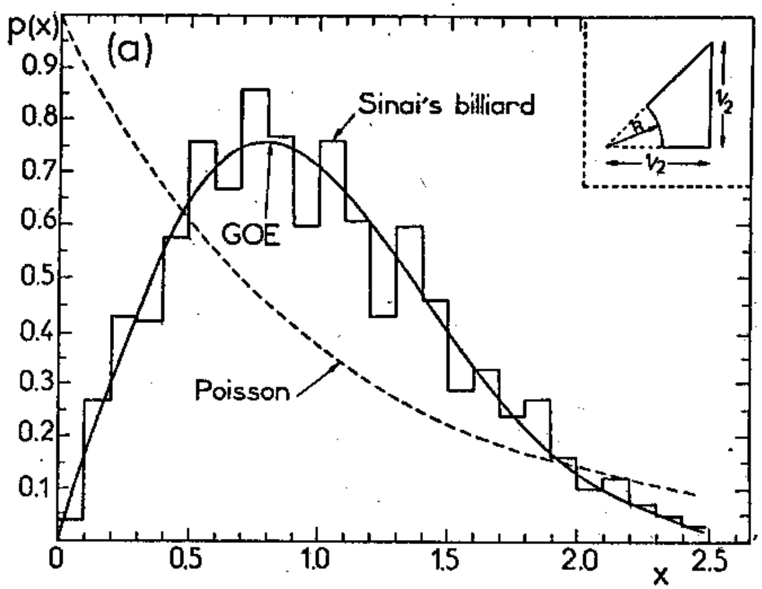

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RMT is not realistic: it assumes many–body forces !

k-body Embedded Gaussian Ensemble

K.K. Mon and J.B. French, Ann. Phys. **95**, 90 (1975)

- n number of particles: **bosons**
- l number of single–particle levels
- k rank of the interaction: $1 \le k \le n$

$$\psi^{\dagger}_{k;\alpha} = \psi^{\dagger}_{j_1,j_2,\ldots,j_k} = \frac{1}{\mathcal{N}_\alpha} \prod_{s=1}^{k} b^{\dagger}_{j_s}, \quad j_1 \le j_2 \le \ldots \le j_k$$

$$\hat{H}^{(\beta)}_k = \sum_{\alpha\gamma} v^{(\beta)}_{k;\alpha\gamma} \psi^{\dagger}_{k;\alpha} \psi_{k;\gamma}, \quad |\mu\rangle = \psi^{\dagger}_{n;\mu}|0\rangle \quad (\beta = 1, 2)$$

$$\boxed{k = n \iff \mathrm{EGE}(\beta) = \mathrm{G}\beta\mathrm{E}}$$

Hilbert space dimension: $N_B = \binom{l+n-1}{n}$, $N_v = \binom{l+k-1}{k}$

Particle number is conserved: $[\hat{n}, \hat{H}^{(\beta)}_k] = 0$

Independent random variables: $K_\beta = \frac{\beta}{2}\binom{l+k-1}{k}\left[\binom{l+k-1}{k} + \delta_{\beta,1}\right]$

Dense limit : $n \to \infty$, l, and k fixed

LB and H.A. Weidenmüller, J. Phys A: Math Gen **36**, 3569 (2003).

k-body Embedded Ensemble for bosons

T Asaga, LB, T Rupp and HA Weidenmüller, Europhys. Lett. **56**, 340 (2001); Ann. Phys. **298** 229 (2002).

The ensemble-averaged second moment:

$$B^{(k,\beta)}_{\mu\nu,\rho\sigma} = \overline{\langle\mu|H_k^{(\beta)}|\sigma\rangle\langle\rho|H_k^{(\beta)}|\nu\rangle} = \frac{1}{N}\sum_{s=0}^{n-k}\sum_a \Lambda_B^{(s)}(k)\left[C^{(sa)}_{\mu\nu}C^{(sa)}_{\rho\sigma} + \delta_{\beta 1}C^{(sa)}_{\mu\rho}C^{(sa)}_{\nu\sigma}\right]$$

$$\Lambda_B^{(s)}(k) = \binom{n-s}{k}\binom{n+s+l-1}{k}, \quad C^{(sa)}_{\mu,\nu} = \langle\mu|\psi^\dagger_{s;\alpha}\psi_{s;\gamma}|\nu\rangle$$

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The fluctuations of the centroids of the spectrum are given by

$$S(k,n,l) \quad = \quad \frac{\overline{[(1/N)\,\mathrm{tr}H_k(\beta)]^2}}{(1/N)\,\mathrm{tr}\overline{[H_k(\beta)]^2}} = \frac{(1+\delta_{\beta 1})\Lambda^{(0)}(n-k)}{\Lambda^{(0)}(k)+\delta_{\beta 1}\sum_s \Lambda^{(s)} d_B^{(s)}},$$

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$$\xrightarrow[n\to\infty]{} \frac{\left(1+\delta_{\beta 1}\right)\binom{2k}{k}\binom{l+k-1}{k}^{-1}}{\binom{2k}{k}+\delta_{\beta 1}\sum_{s=0}^{k}\binom{2k}{k+s}\binom{l+k+s-1}{k+s}^{-1}d_B^{(s)}} \neq 0$$

The bEGE is **non-ergodic** in the **dense limit**

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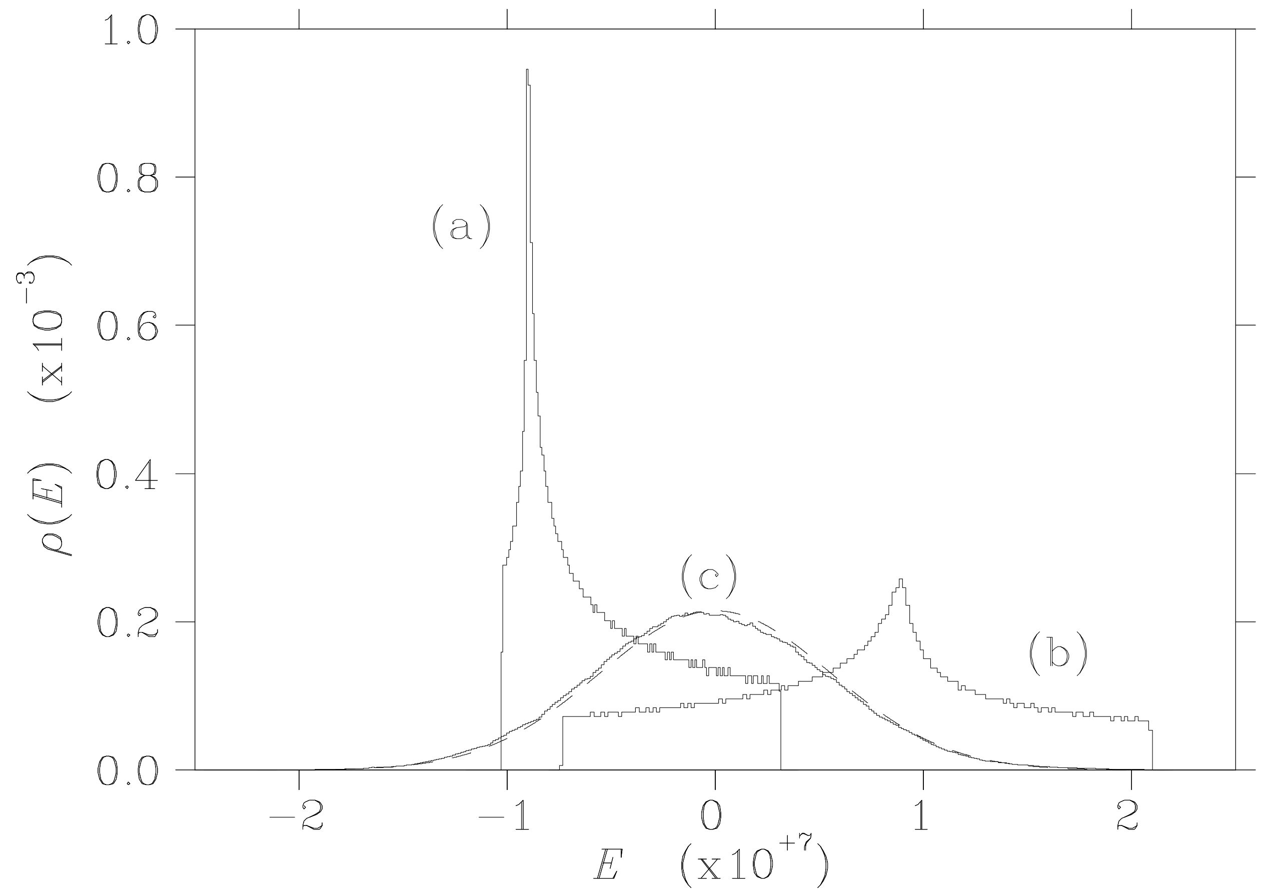
Non-ergodic behaviour of the ensemble



Spectral density $k = l = 2$

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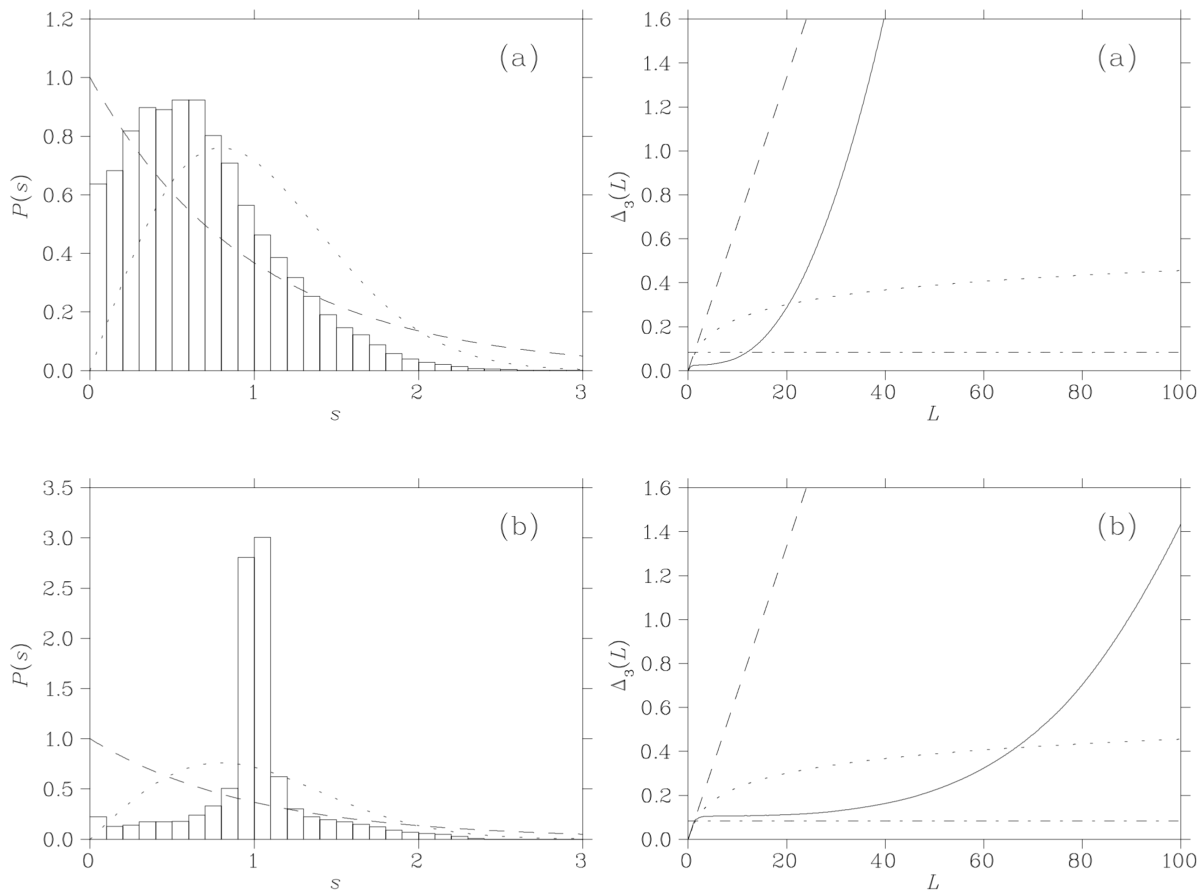

(a) Ensemble unfolding and (b) spectral unfolding $k = l = 2$

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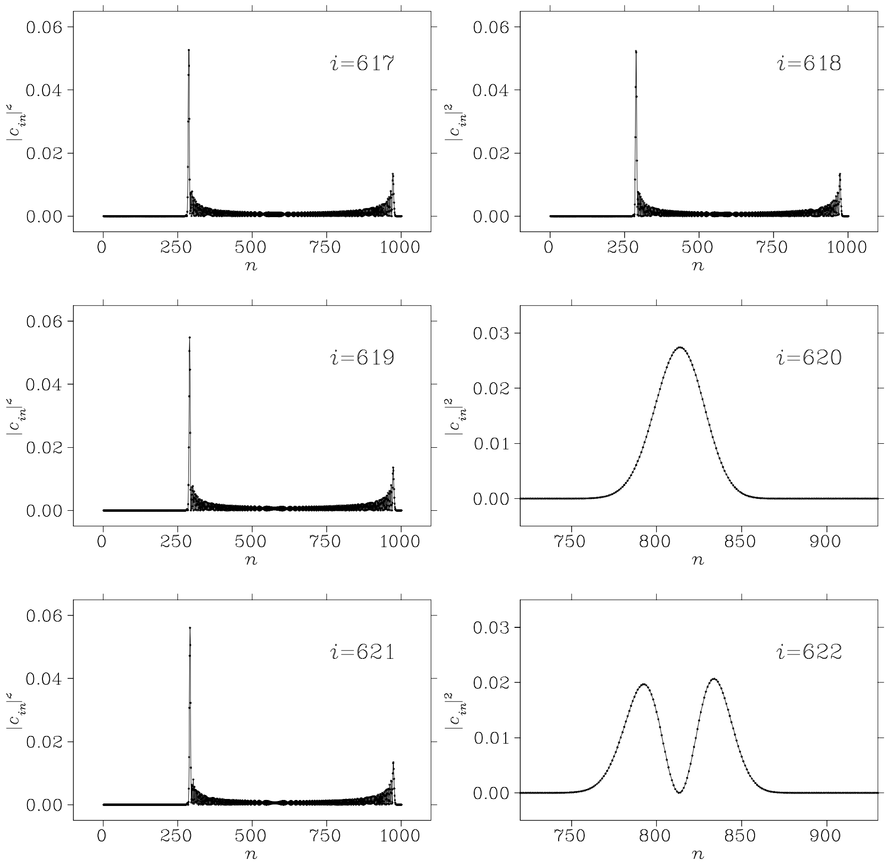
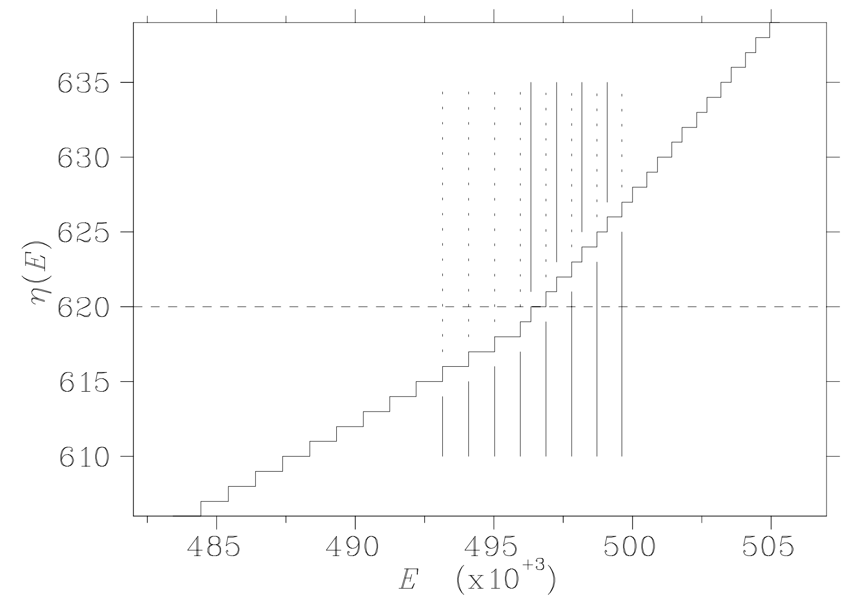




Eigenfunctions $k = l = 2$

Nearest-neighbor statistics for $l = 2$

n	number of **bosons**
$l = 2$	two single–particle levels $\Rightarrow N = n + 1$
k	rank of the interaction: $1 \leq k \leq n$

$$\hat{H}_k^{(\beta)} = \sum_{0 \leq s,t \leq k} v_{s,t}^{(\beta)} \frac{(\hat{b}_1^\dagger)^s (\hat{b}_2^\dagger)^{k-s} (\hat{b}_1)^t (\hat{b}_2)^{k-t}}{[s!(k-s)!t!(k-t)!]^{1/2}}, \qquad |\mu\rangle \equiv \frac{(\hat{b}_1^\dagger)^\mu (\hat{b}_2^\dagger)^{n-\mu}}{[\mu!(n-\mu)!]^{1/2}} |0\rangle$$

$$\text{Hilbert space dimension:} \qquad N_B = n + 1$$

$$\text{Independent random variables:} \qquad K_\beta = \frac{\beta}{2}(k+1)[k+1+\delta_{\beta,1}]$$

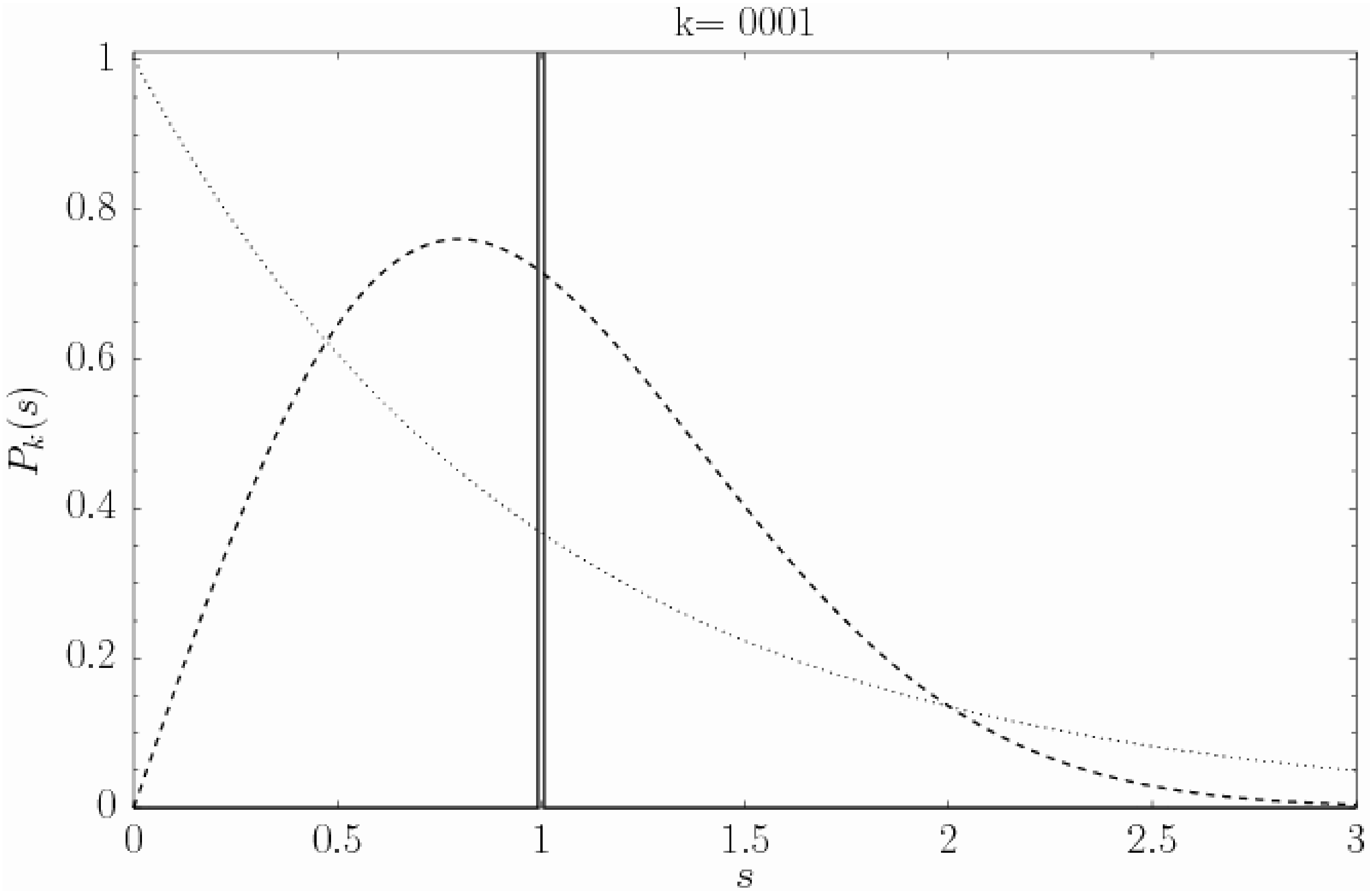
Transition in the spectral correlations of the ensemble in terms of k:

(i) For $k = 1$ we have the superposition of $l = 2$ independent harmonic–oscillator spectra

(ii) For $k = n$ we have exactly RMT results (by definition)

Nearest-neighbor statistics for $l = 2$

S. Hernández-Quiroz and LB, Phys. Rev. E **81**, 036218 (2010).



$v \in$GUE(k+1) (Movie2)

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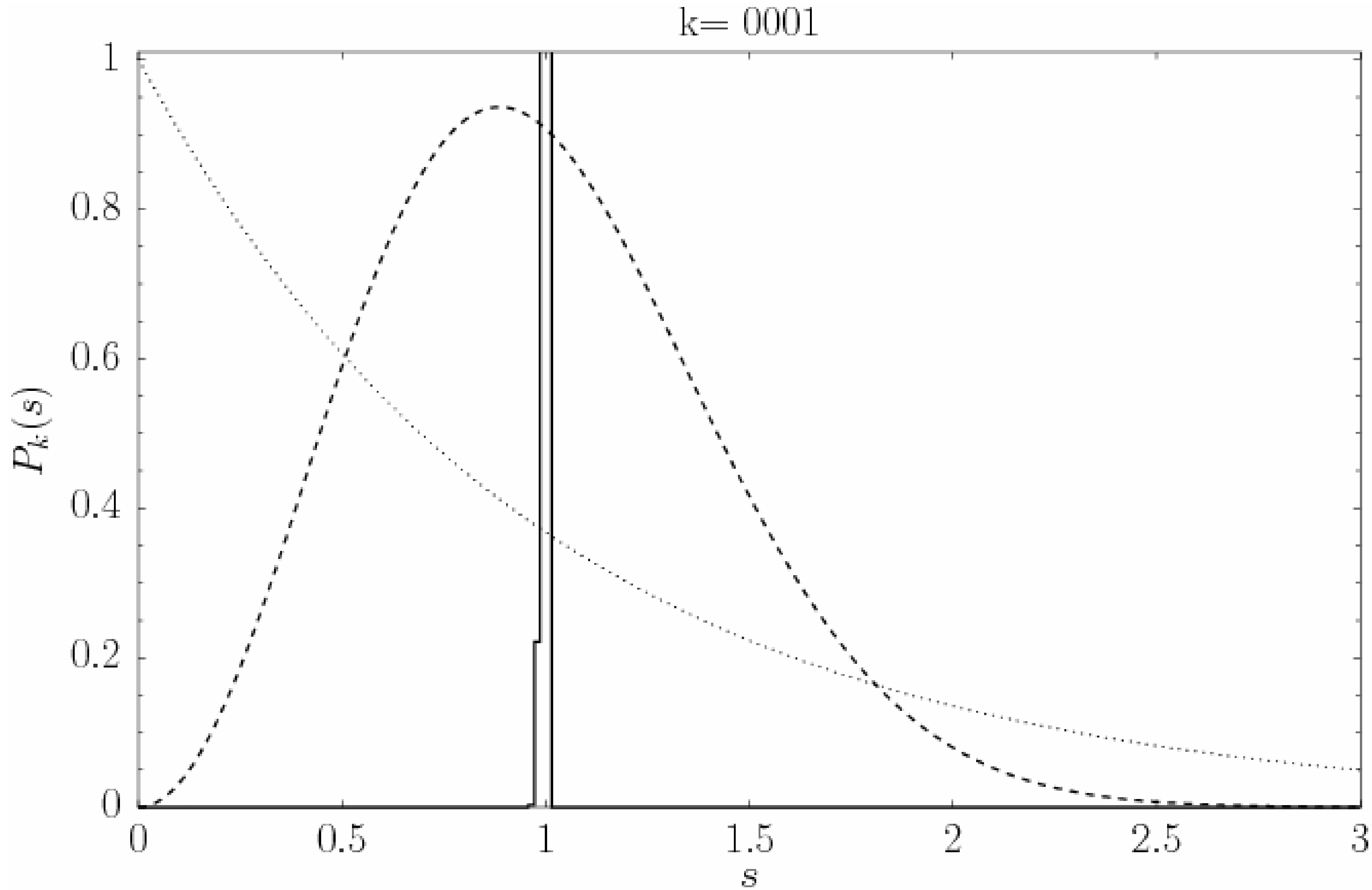

$v \in$GOE(k+1) (Movie1)

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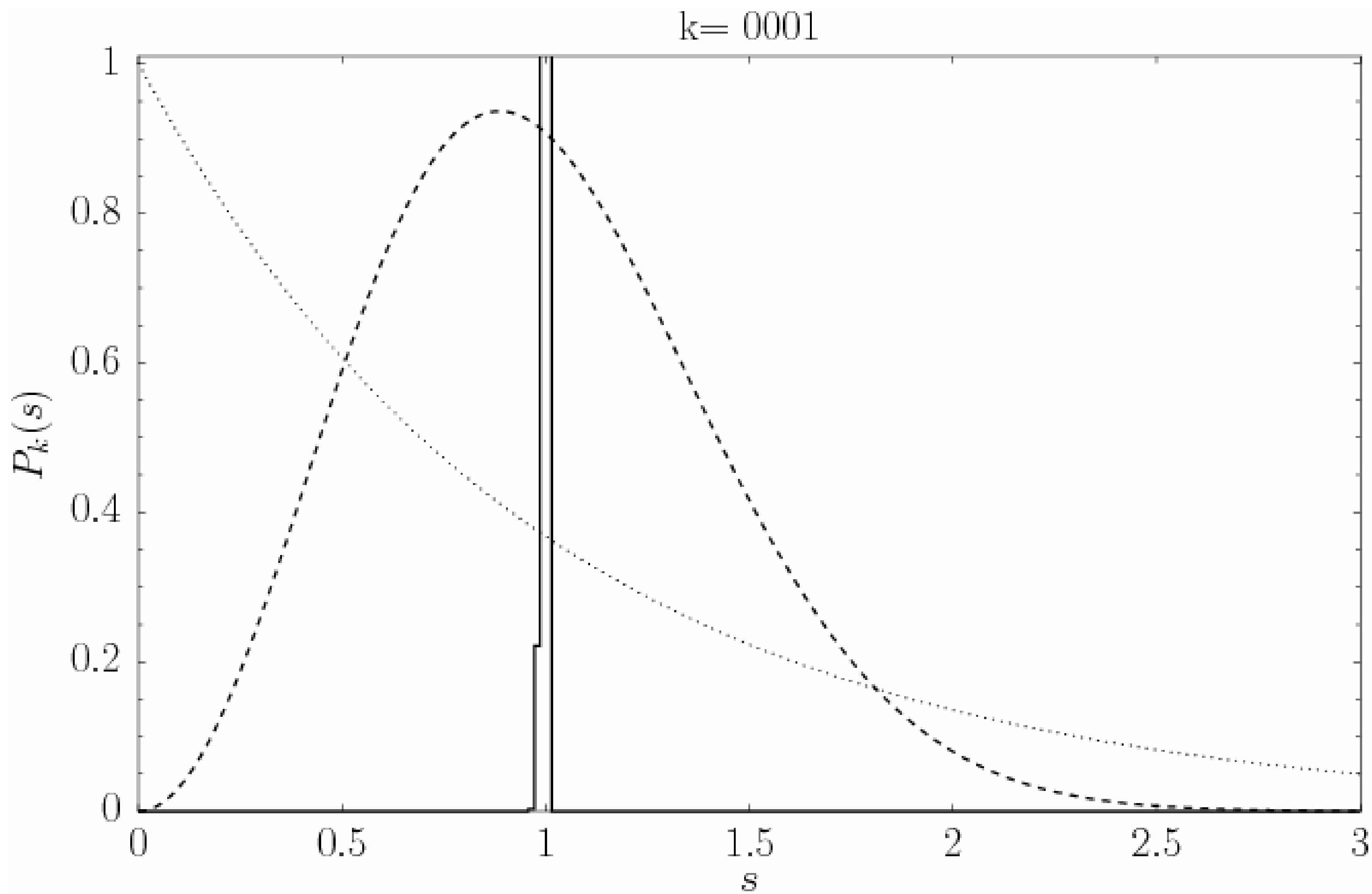
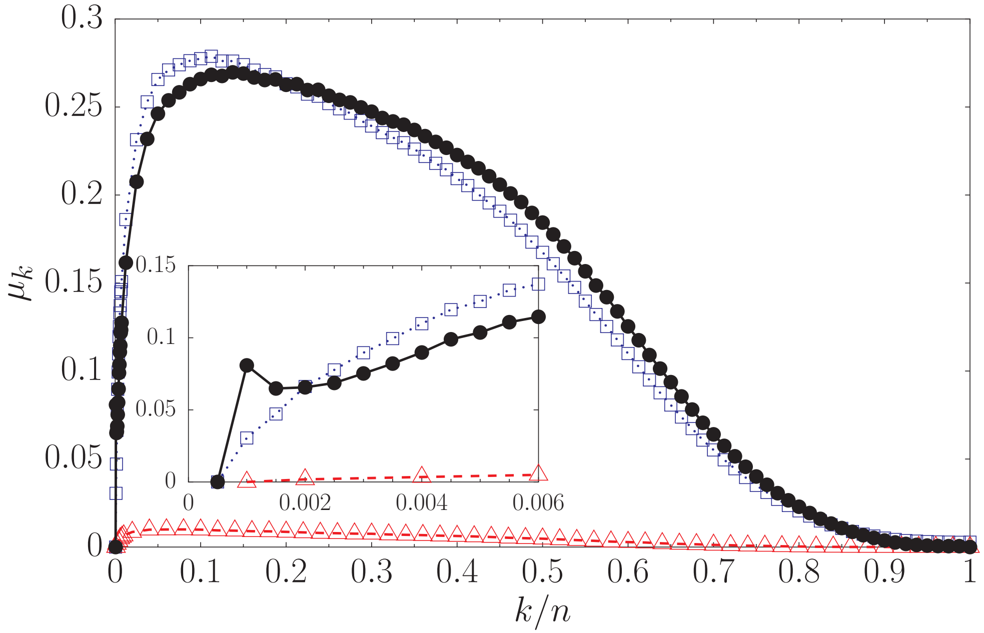


$v \in$GOE(k+1)

The peak at $s = 0$ reflects the time reversal invariance

Nearest-neighbor statistics for $l = 2$



Shnirelman doublets: states related by time-reversal symmetry in a quantized 2-dof integrable system

Semiclassical limit

LB, C. Jung and F. Leyvraz, J. Phys A: Math and Gen **36**, L217 (2003).

For $l = 2$

$$\hat{H}_k^{(\beta)} = \sum_{s,t=0}^{k} v_{s,t}^{(\beta)} \frac{(\hat{b}_1^\dagger)^s (\hat{b}_2^\dagger)^{k-s} (\hat{b}_1)^t (\hat{b}_2)^{k-t}}{[s!(k-s)!t!(k-t)!]^{1/2}}$$

Semiclassical limit: $n \to \infty$

(i) Symmetrize $\hat{H}_k$: $\hat{b}_r^\dagger \hat{b}_s = (\hat{b}_r^\dagger \hat{b}_s + \hat{b}_s \hat{b}_r^\dagger - \delta_{r,s})/2$

(ii) Heisenberg semiclassical rules: $b_r \to I_r^{1/2} \exp[i\phi_r]$
Time-reversal invariance $\beta = 1$: $\phi_r \to -\phi_r$

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(iii) The Hamiltonian $\mathcal{H}_k$ has two degrees of freedom $(l = 2)$

(iv) Angles appear only as $\phi_1 - \phi_2$

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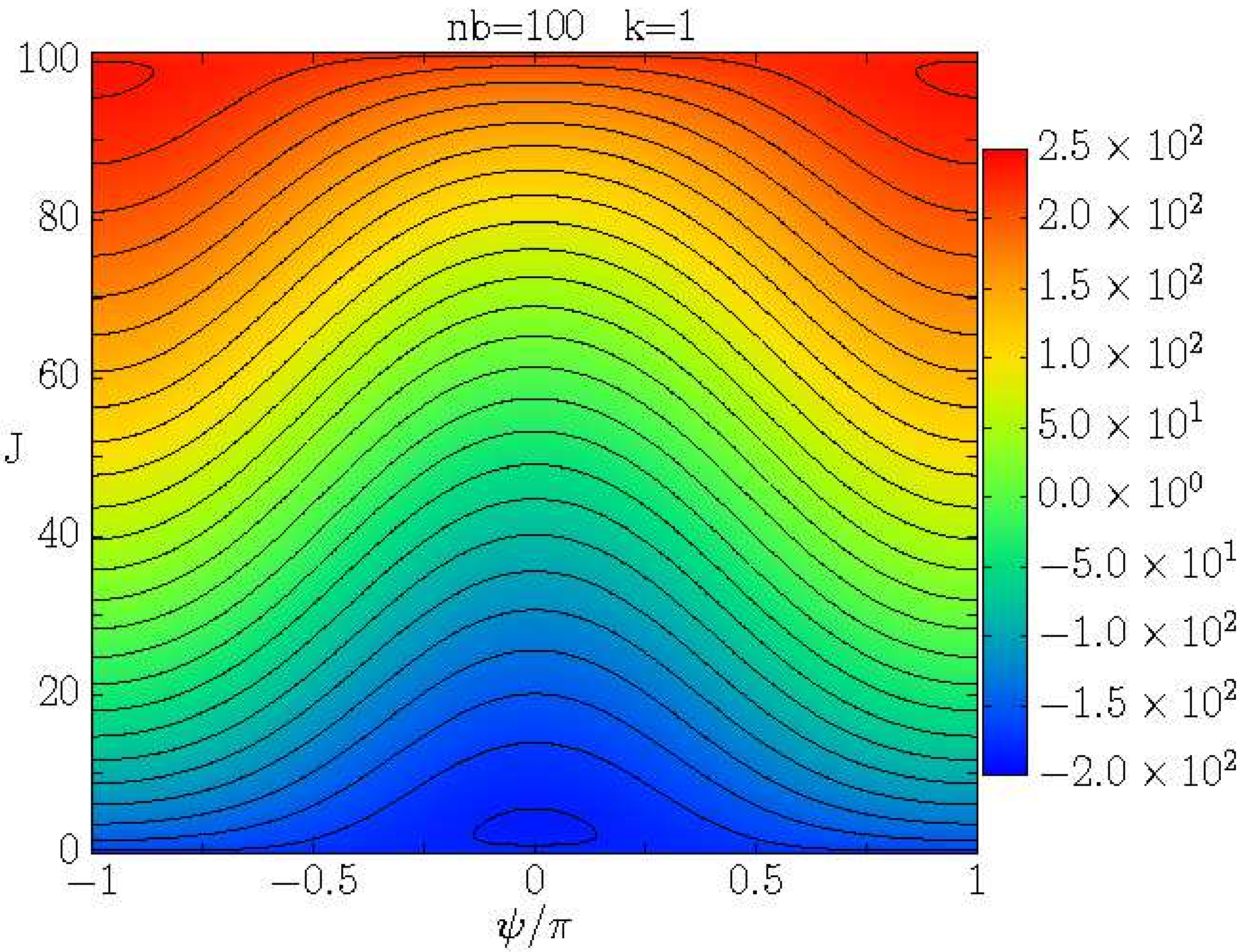
Therefore, the classical Hamiltonian $\mathcal{H}_k$ is Liouville integrable

Semiclassical limit

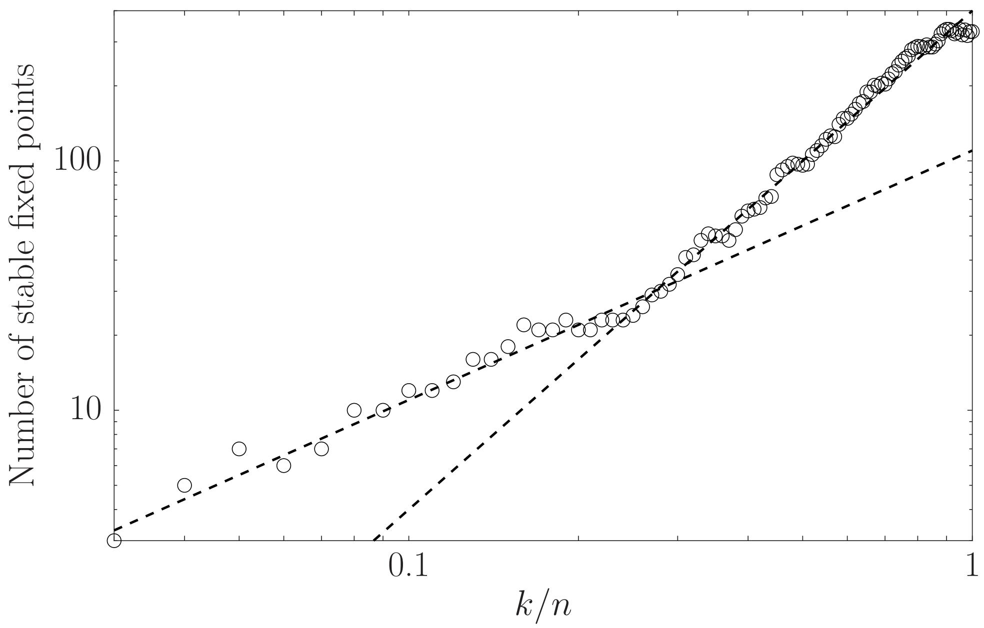
S. Hernández-Quiroz and LB, Phys. Rev. E **81**, 036218 (2010).



(Movie3)

Semiclassical limit

S. Hernández-Quiroz and LB, Phys. Rev. E **81**, 036218 (2010).



To understand this, we note:

(i) The number of equilibrium points grows as $\sim k^2$, for k large enough

(ii) The phase space volume is $N = n + 1$

Then, the effective phase-space volume around elliptic points shrinks, so EBK tori must occupy more extended regions in phase space

Summary

- The bEGβE is **non-ergodic** in the **dense limit**
- These results carry over to more general bosonic hamiltonians $H = H_0 + V$ in the dense limit, e.g., the Bose-Hubbard model.
- Each member of the bEGβE for $l = 2$ is **Liouville integrable** in the semi-classical limit ($n \to \infty$)
- For $\beta = 1$ and k fixed, there is a subsequence of almost-degenerate levels (Shnirelman doublets), in accordance to a theorem by Shnirelman [1974]. The number of Shnirelman doublets depends on k, and vanishes as $k \to n$, as ergodicity reappears.
- Preliminary results indicate that the survival probability of initial eigenstates (of the *unperturbed* Hamiltonian) display a (correlation) hole.

Thank you!