LOCALIZATION MEASURES IN PHASE SPACE

Dicke-Team Annual Gathering

David Villaseñor

ICN-UNAM

February, 2021

Dicke Team

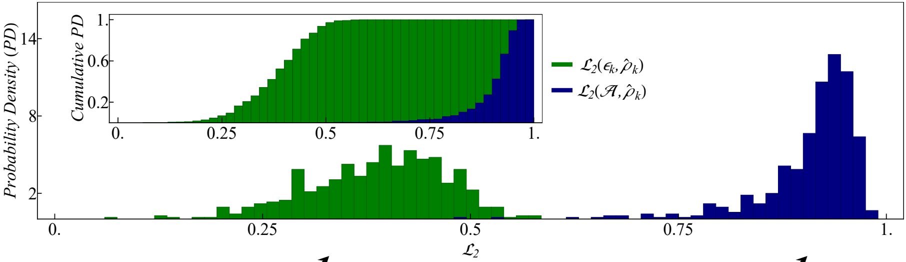
- Jorge G. Hirsch, ICN-UNAM, CDMX.
- Lea F. Santos, Yeshiva University, New York, USA.
- Sergio Lerma-Hernández, UV, Xalapa, Veracruz.
- Miguel A. Bastarrachea-Magnani, UAM-Iztapalapa, CDMX.
- Saúl Pilatowsky-Cameo, ICN-UNAM, CDMX.
- Eduardo J. Torres-Herrera, BUAP, Puebla, Puebla.
- Jorge Chávez-Carlos, ICF-UNAM, Cuernavaca, Morelos.

Outline

Objective

Two localization measures give different interpretations!

Why this happens? We want to understand these results.



Measure 1: localized states?

Measure 2: delocalized states?

Introduction

A localization measure quantifies the occupation degree of a quantum state in a discrete or continuous basis.

A well-known example is the participation ratio, which measures the localization of a state $|\Psi\rangle$ over an N-dimensional discrete basis $\{|\phi_k\rangle\}$ of some Hilbert space $\mathcal{H}$

$$P_R = \left(\sum_{k=1}^{N} p_k^2 \right)^{-1}, \qquad P_R \in [1, N], \tag{1}$$

where $p_k = |\langle\phi_k|\Psi\rangle|^2$ is the probability to find the state $|\Psi\rangle$ in each $|\phi_k\rangle$ basis state. $P_R = 1$ defines a maximum of localization when one single basis state $|\phi_k\rangle$ localizes the state $|\Psi\rangle$. Moreover, $P_R = N$ defines a maximum of delocalization when all basis states localize equally $|\langle\phi_k|\Psi\rangle| = 1/\sqrt{N}$ the state $|\Psi\rangle$.

Localization Measures in Discrete Spaces

Consider a set Ω with M disjoint elements $\xi_k \in \Omega$. An arbitrary state $|\Upsilon\rangle$ has probabilities $p_k = |\langle\xi_k|\Upsilon\rangle|^2$ to be in each element ξ_k, with total probability $\sum_{k=1}^{M} p_k = 1$. We are looking for a measure to estimate the number of elements contributing to localize the state $|\Upsilon\rangle$. If the state $|\Upsilon\rangle$ is localized in a subset $\omega \subset \Omega$ with dimension $m \leq M$, the maximal participation is obtained when

$$p_k = \begin{cases} 1/m & \xi_k \in \omega \\ 0 & \text{else} \end{cases} . \tag{2}$$

Now, if we consider the weighted mean $G = \sum_{k=1}^{M} p_k g(p_k)$ of a given function $g(x)$, whose inverse function is $g^{-1}(x)$, such that $g^{-1}(g(x)) = x$; then, the localization measure we are looking for is

$$L = \frac{1}{g^{-1}(G)}, \tag{3}$$

and the maximal participation is the dimension m of the subset ω

$$L_{\text{max}}^{\omega} = m. \tag{4}$$

Localization Measures in Discrete Spaces

Particular cases of the localization measure L are:

- Shannon Entropy (SE): $g(x) = \ln(x)$, $g^{-1}(x) = e^x$,
  $G = \sum_{k=1}^{M} p_k \ln(p_k)$,

$$L = e^{-G} = exp\left(-\sum_{k=1}^{M} p_k \ln(p_k)\right) = e^{H_{\mathrm{SE}}}, \qquad H_{\mathrm{SE}} = -G. \quad (5)$$

- Generalized Renyi Entropy (GRE): $g(x) = x^n$, $g^{-1}(x) = x^{1/n}$,
  $G = \sum_{k=1}^{M} p_k^{n+1}$,

$$L = G^{-1/n} = \left(\sum_{k=1}^{M} p_k^{n+1}\right)^{-1/n} = e^{H_{\mathrm{GRE}}}, \qquad H_{\mathrm{GRE}} = -\frac{1}{n}\ln(G). \quad (6)$$

  This case gives the participation ratio, $L = \left(\sum_{k=1}^{M} p_k^2\right)^{-1}$, when $n = 1$.

Localization Measures in Continuous Spaces

Now consider a basis $\Omega(k)$ with parameter $k \in \mathcal{M}$ defined in the continuous space $\mathcal{M}$, such that an arbitrary state $|\Upsilon\rangle$ has probability density $p(k) = |\langle k|\Upsilon\rangle|^2$ to be localized in $\mathcal{M}$, with total probability $\int_{\mathcal{M}} dk\, p(k) = 1$. Analogously to a discrete space, if the state $|\Upsilon\rangle$ is localized in a subspace $\mathcal{M}_m \subset \mathcal{M}$, the maximal participation is obtained when

$$p(k) = \begin{cases} 1/V_m & k \in \mathcal{M}_m \\ 0 & \text{else} \end{cases}, \qquad V_m = \int_{\mathcal{M}_m} dk, \tag{7}$$

and it is equal to the volume V_m of the subspace $\mathcal{M}_m$

$$L_{\max}^{\mathcal{M}_m} = V_m = \int_{\mathcal{M}_m} dk. \tag{8}$$

Localization Measures in Continuous Spaces

The generalization to continuous basis is summarized in the next table:

	Set Ω with dimension M	Basis $\Omega(k)$ with $k \in \mathcal{M}$
	Subset $\omega \subset \Omega$ with $m \leq M$	Subspace $\mathcal{M}_m \subset \mathcal{M}$
L_{SE}	$exp\left(-\sum_{k=1}^{M} p_k \ln(p_k)\right)$	$exp\left(-\int_{\mathcal{M}} dk\, p(k) \ln[p(k)]\right)$
L_{GRE}	$\left(\sum_{k=1}^{M} p_k^{n+1}\right)^{-1/n}$	$\left(\int_{\mathcal{M}} dk\, p^{n+1}(k)\right)^{-1/n}$
P_R	$\left(\sum_{k=1}^{M} p_k^2\right)^{-1}$	$\left(\int_{\mathcal{M}} dk\, p^2(k)\right)^{-1}$
$L_{\max}$	m	$V_m = \int_{\mathcal{M}_m} dk$
SE: Shannon Entropy, GRE: Generalized Renyi Entropy		

Relative Localization Measures

When the subspace $\mathcal{M}_m$ is unbounded, the next issue arises

$$V_m = \int_{\mathcal{M}_m} dk \to \infty, \tag{9}$$

that is, the volume of the subspace is infinite and the distribution $p(k)$ is arbitrarily delocalized.

To avoid this issue, in our studies the localization measures are defined in bounded spaces only. They are divided by the volume of the bounded space (maximal localization), such that we can have a relative localization measure

$$L \to \frac{L}{V_m} \in (0, 1]. \tag{10}$$

Toy Model: Dicke Model

The Dicke model is given by the Hamiltonian

$$\hat{H}_D = \omega \hat{a}^\dagger \hat{a} + \omega_0 \hat{J}_z + \frac{2\gamma}{\sqrt{\mathcal{N}}} \hat{J}_x (\hat{a}^\dagger + \hat{a}), \tag{11}$$

where:

- $\hat{a}$ ($\hat{a}^\dagger$) is the annihilation (creation) operator.
- $\hat{J}_{x,y,z} = \sum_{n=1}^{\mathcal{N}} \hat{\sigma}^n_{x,y,z}/2$ are the collective pseudo-spin operators, and $\hat{\sigma}^n_{x,y,z}$ are the Pauli matrices.
- $\mathcal{N}$ is the total number of atoms within the system.
- ω is the radiation frequency of the electromagnetic field.
- ω_0 is the transition frequency between two atomic levels.
- γ is the coupling strength, whose critical value $\gamma_c = \sqrt{\omega\omega_0}/2$ separates two phases: normal ($\gamma < \gamma_c$) and super-radiant ($\gamma > \gamma_c$).

Classical Dicke Model

The classical Dicke Hamiltonian can be obtained by taking the expectation value of $\hat{H}_D$ under Glauber $|q,p\rangle = \hat{D}(q,p)|0\rangle$ and Bloch $|Q,P\rangle = \hat{R}(Q,P)|j,-j\rangle$ coherent states, and dividing it by the system size $j = \mathcal{N}/2$

$$
\begin{aligned}
h_{\text{cl}}(\mathbf{x}) =& \frac{\langle q,p| \otimes \langle Q,P|\hat{H}_D|q,p\rangle \otimes |Q,P\rangle}{j} \\
=& \frac{\omega}{2}\left(q^2+p^2\right) + \frac{\omega_0}{2}\left(Q^2+P^2\right) + 2\gamma\sqrt{1-\frac{Q^2+P^2}{4}}Qq - \omega_0,
\end{aligned} \tag{12}
$$

which gives a 4-dimensional phase space $\mathcal{D} = (q,p;Q,P)$ independent of this size.

Localization Measure over an Energy Shell

A localization measure over the subspace $\mathcal{D}_\epsilon$ of a single energy shell $\epsilon = E/j$ of the available phase space $\mathcal{D}$ of the Dicke model is defined as

$$\mathfrak{L}(\epsilon,\hat{\rho}) = \frac{C_\epsilon^2}{\mathcal{V}(\epsilon)}\left(\int_{\mathcal{D}_\epsilon} d\mathbf{s}\mathcal{Q}_{\hat{\rho}}^2(\mathbf{x})\right)^{-1}, \qquad C_\epsilon = \int_{\mathcal{D}_\epsilon} d\mathbf{s}\mathcal{Q}_{\hat{\rho}}(\mathbf{x}), \tag{13}$$

where C_ϵ ensures normalization, $\mathcal{Q}_{\hat{\rho}}(\mathbf{x}) = \langle \mathbf{x}|\hat{\rho}|\mathbf{x}\rangle$ is the Husimi function of the state $\hat{\rho}$ in the coherent states basis $\{|\mathbf{x}\rangle|\mathbf{x} = (q,p;Q,P)\}$, $d\mathbf{s} = \delta(h_{\text{cl}}(\mathbf{x}) - \epsilon)d\mathbf{x}$, and the subspace is $\mathcal{D}_\epsilon = \{\mathbf{x} = (q,p;Q,P)|h_{\text{cl}}(\mathbf{x}) = \epsilon\}$. We divide the measure additionally by the volume of the subspace $\mathcal{V}(\epsilon) = \int_{\mathcal{D}_\epsilon} d\mathbf{s}$, such that, this defines a relative measure bounded in the interval $\mathfrak{L} \in (0,1]$.

See: S. Pilatowsky-Cameo et al., Nat. Comm., (2021), in press.

Validity of the Energy-Shell Localization Measure

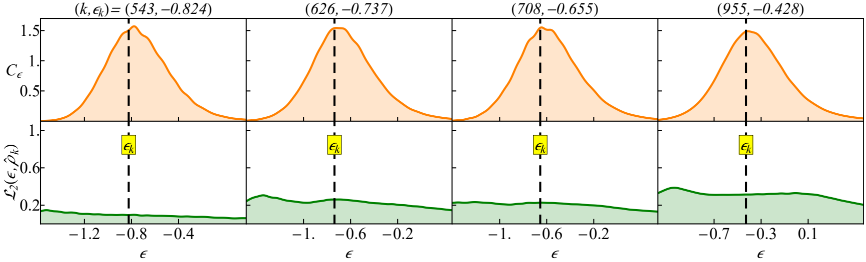


Figure: **Top panels:** Energy distribution of the Husimi function C_ϵ for selected eigenstates $\hat{\rho}_k$ located at the chaotic energy region $\epsilon_k \in (-0.83, -0.43)$. **Bottom panels:** Energy distribution of the localization measure $\mathfrak{L}(\epsilon, \hat{\rho}_k)$ for the same eigenstates.

Evolved States and Time Average

The time evolution of the energy-shell localization measure $\mathfrak{L}(\epsilon, \hat{\rho}(t))$ can be obtained using evolved states

$$\hat{\rho}(t) = e^{-i\hat{H}_D t}\hat{\rho}e^{i\hat{H}_D t}, \tag{14}$$

and later we can compute the measure $\mathfrak{L}(\epsilon, \bar{\rho})$ for a mixed state $\bar{\rho}$ averaged in time

$$\bar{\rho} = \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\hat{\rho}(t). \tag{15}$$

We will see that the last step is very important, since this will allow us to reach **quantum ergodicity** for some kind of states.

Time-Averaged States and Quantum Ergodicity

See: S. Pilatowsky-Cameo et al., Nat. Comm., (2021), in press.

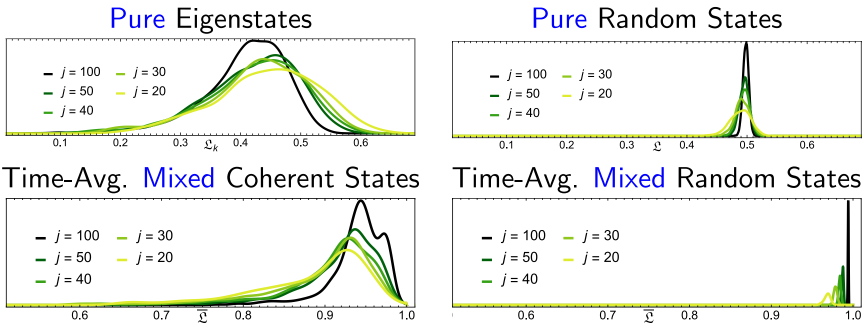


Figure: **Top panels:** Localization measure $\mathfrak{L}(\epsilon_k, \hat{\rho}_k)$ ($\mathfrak{L}(\epsilon, \hat{\rho}_R)$) for eigenstates $\hat{\rho}_k$ (random states $\hat{\rho}_R$) located at the chaotic energy region $\epsilon_k \in (-0.8, 0)$ ($\epsilon = -0.5$). **Bottom panels:** Localization measure $\mathfrak{L}(\epsilon, \bar{\rho}_{CS})$ ($\mathfrak{L}(\epsilon, \bar{\rho}_R)$) for mixed coherent states $\bar{\rho}_{CS}$ (random states $\bar{\rho}_R$) averaged in time, located at $\epsilon = -0.5$.

Localization Measure over the Atomic Subspace

An alternative localization measure was also studied in the Dicke model. This measure was defined in the atomic subspace of the model

$$\mathfrak{L}(\mathcal{A},\hat{\rho}) = \frac{C^2}{4\pi}\left(\int_{\mathcal{A}} dQdP\,\widetilde{\mathcal{Q}}^2_{\hat{\rho}}(Q,P)\right)^{-1}, \qquad C = \frac{2\pi}{j}\frac{4\pi}{2j+1}, \tag{16}$$

where C ensures normalization, $\widetilde{\mathcal{Q}}_{\hat{\rho}}(Q,P) = \int dqdp\,\mathcal{Q}_{\hat{\rho}}(q,p;Q,P)$ is the Husimi function of the state $\hat{\rho}$ projected in the atomic variables (Q,P), and the subspace is $\mathcal{A} = \{(Q,P)|Q^2+P^2 \leq 4\}$. This measure was also defined as a relative measure, $\mathfrak{L} \in (0,1]$, where the volume of the subspace is $\mathcal{V}(\mathcal{A}) = \int_{\mathcal{A}} dQdP = 4\pi$.

See: Q. Wang and M. Robnik, Phys. Rev. E **102**, 032212 (2020).

Differences between Localization Measures

The differences between both measures $\mathfrak{L}(\epsilon, \hat{\rho})$ and $\mathfrak{L}(\mathcal{A}, \hat{\rho})$ are summarized in the next table:

	Energy Shell, $\mathfrak{L}(\epsilon, \hat{\rho})$	Atomic Subspace, $\mathfrak{L}(\mathcal{A}, \hat{\rho})$
S	$\mathcal{D}_\epsilon = \{\mathbf{x} = (q, p; Q, P) \vert h_{\mathrm{cl}}(\mathbf{x}) = \epsilon\}$	$\mathcal{A} = \{(Q, P) \vert Q^2 + P^2 \leq 4\}$
D	$\frac{1}{C_\epsilon} \mathcal{Q}_{\hat{\rho}}(\mathbf{x})$	$\frac{1}{C} \int dq dp \mathcal{Q}_{\hat{\rho}}(q, p; Q, P)$
dV	$d\mathbf{s} = \delta(h_{\mathrm{cl}}(\mathbf{x}) - \epsilon) d\mathbf{x}$	$dQdP$
V	$\mathcal{V}(\epsilon) = \int_{\mathcal{D}_\epsilon} d\mathbf{s}$	$\mathcal{V}(\mathcal{A}) = \int_{\mathcal{A}} dQdP = 4\pi$

S: Subspace, D: Distribution, dV: Volume Element, V: Volume

Localization Measures for Eigenstates

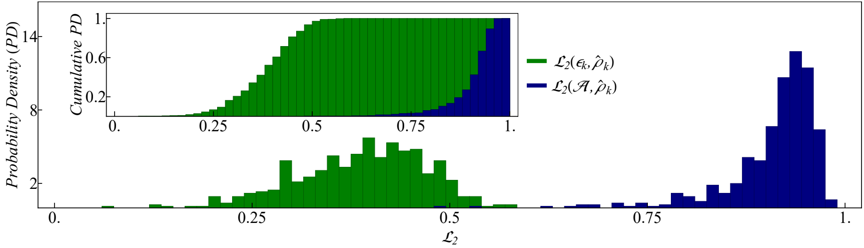


Figure: Statistical distribution of localization measures $\mathfrak{L}(\epsilon_k, \hat{\rho}_k)$ and $\mathfrak{L}(\mathcal{A}, \hat{\rho}_k)$ for 501 eigenstates $\hat{\rho}_k$ located at the chaotic energy region $\epsilon_k \in (1, 1.274)$. **Inset:** Cumulative distribution for the same eigenstates.

Localization Measures for Coherent States

Evolved Pure State $\hat{\rho}_x$ at t Time-Avg. Mixed State $\bar{\rho}_x$ at $(0, T)$

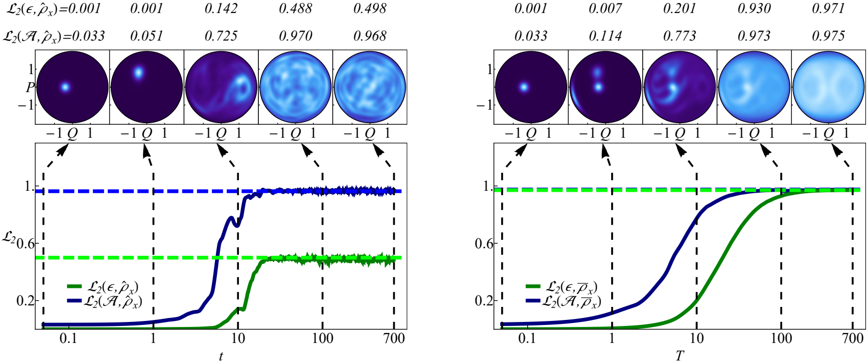
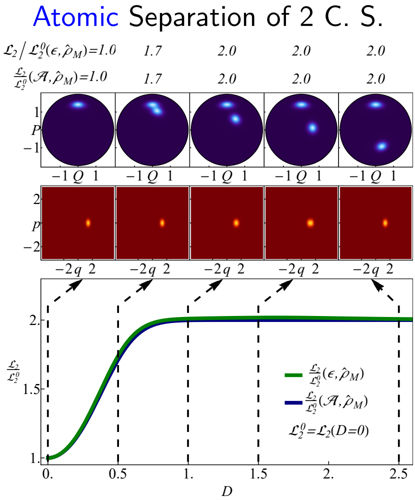


Figure: Localization measures $\mathfrak{L}(\epsilon, \hat{\rho}_x)$ and $\mathfrak{L}(\mathcal{A}, \hat{\rho}_x)$ ($\mathfrak{L}(\epsilon, \bar{\rho}_x)$ and $\mathfrak{L}(\mathcal{A}, \bar{\rho}_x)$) for an evolved pure (time-averaged mixed) coherent state $\hat{\rho}_x$ at t ($\bar{\rho}_x$ at $(0, T)$), located at the chaotic energy region $\epsilon = 1$.

Localization Measures for Coherent States



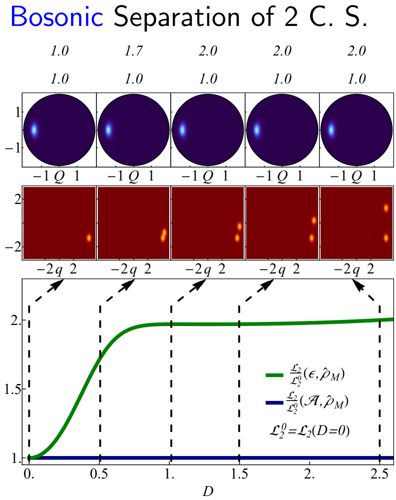


Figure: Localization measures $\mathfrak{L}(\epsilon, \hat{\rho}_M)$ and $\mathfrak{L}(\mathcal{A}, \hat{\rho}_M)$ for mixed coherent states $\hat{\rho}_M$ separated by a distance D in phase space, located at the chaotic energy region $\epsilon = 1$.

Localization Measures for Coherent States

Saturation of Atomic Plane (Q, P) for n Mixed Coherent States

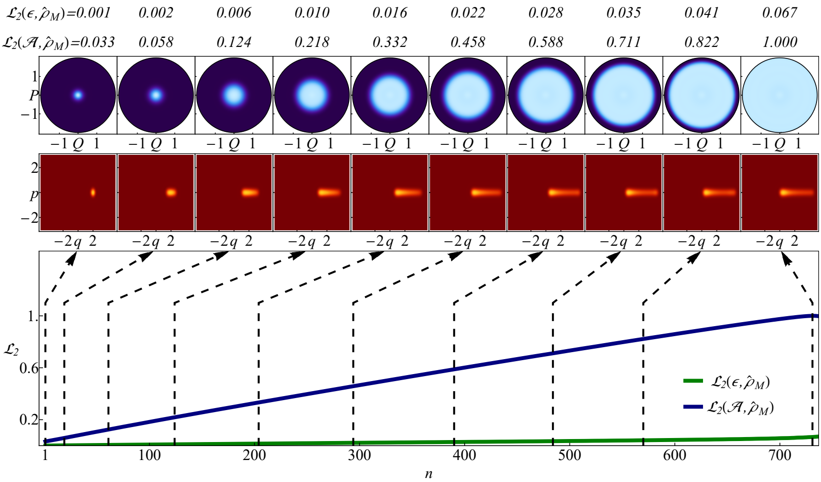


Figure: Localization measures $\mathfrak{L}(\epsilon, \hat{\rho}_M)$ and $\mathfrak{L}(\mathcal{A}, \hat{\rho}_M)$ for coherent states $\hat{\rho}_M$ mixed in the atomic plane (Q, P), located at the chaotic energy region $\epsilon = 1$.

Conclusions

- There is no universal way to define a relative localization measure ($L \in (0, 1]$) of quantum states in an unbounded phase space.
- The use of a bounded space is needed to define the relative localization measure appropriately.
- The selection of the bounded space is very important and has to be done carefully if we want to obtain enough information about localization of the states.