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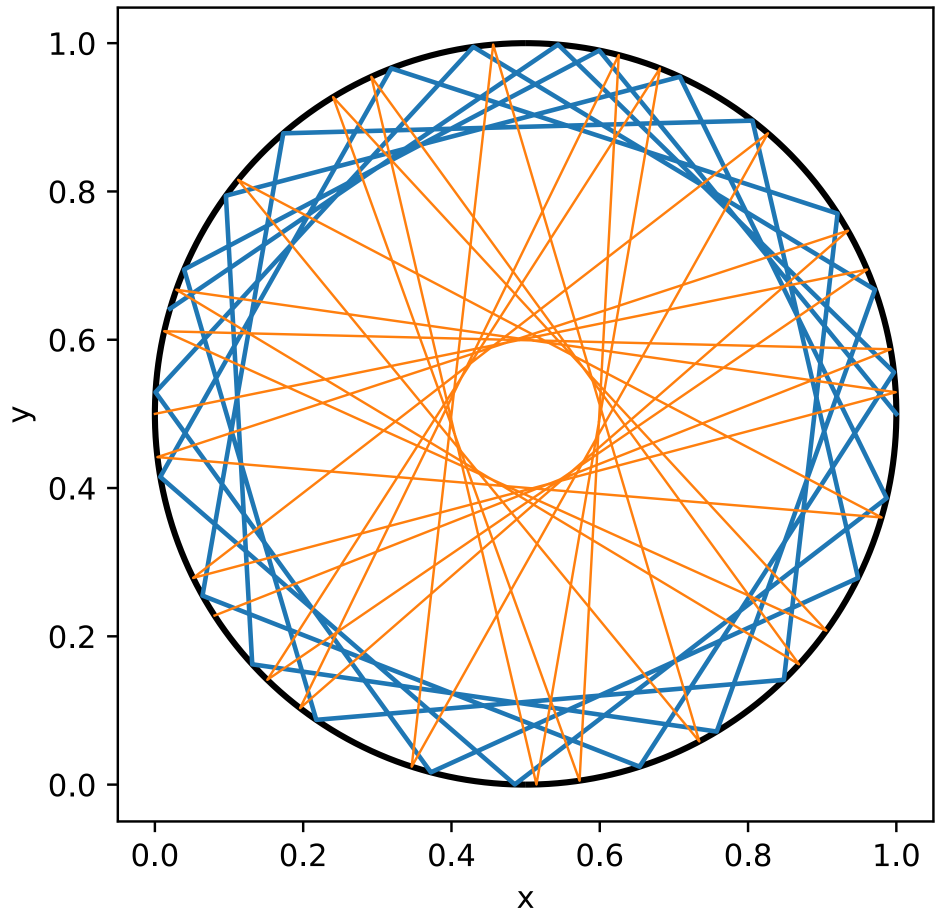
LYAPUNOV EXPONENTS IN CLASSICAL 2D BILLIARDS

A DYNAMICAL BILLIARD

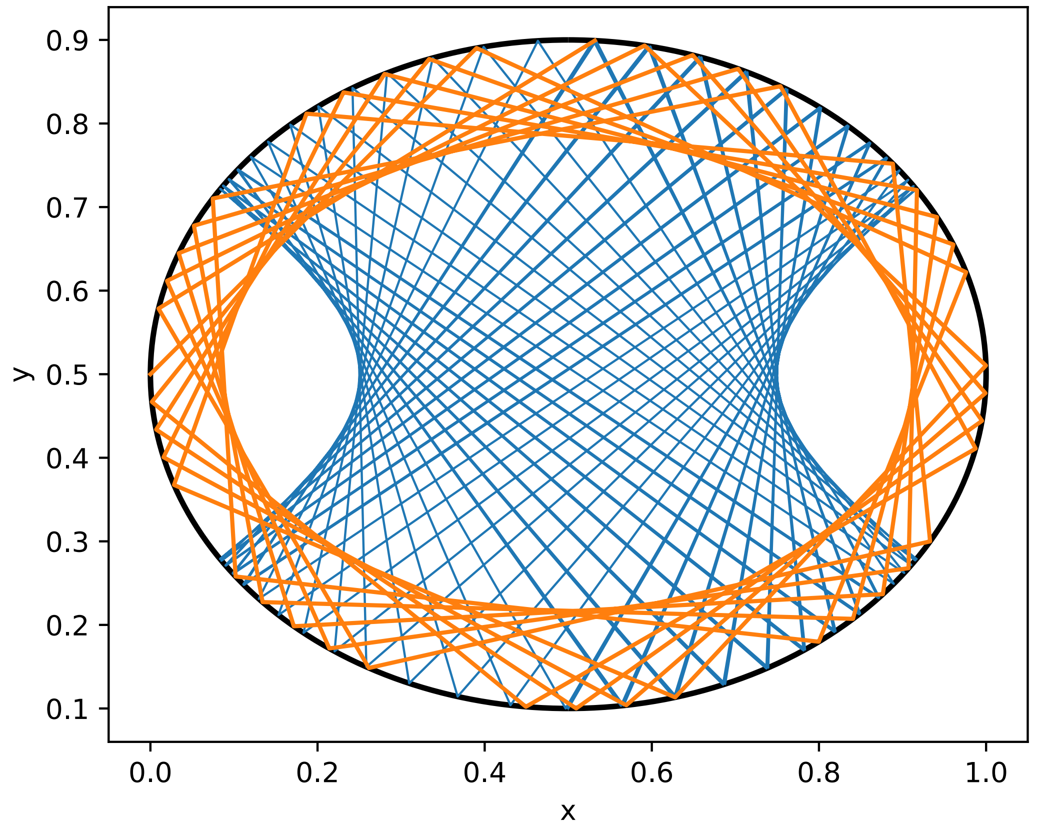
- We define a billiards table as a set of obstacles (boundaries).
- For a 2D billiard particles evolve along a free path on the plane.
- Every particle collision results in an specular reflection.
- We suppose $|\mathbf{p}| = 1$ and unitary mass.

TYPICAL BILLIARDS

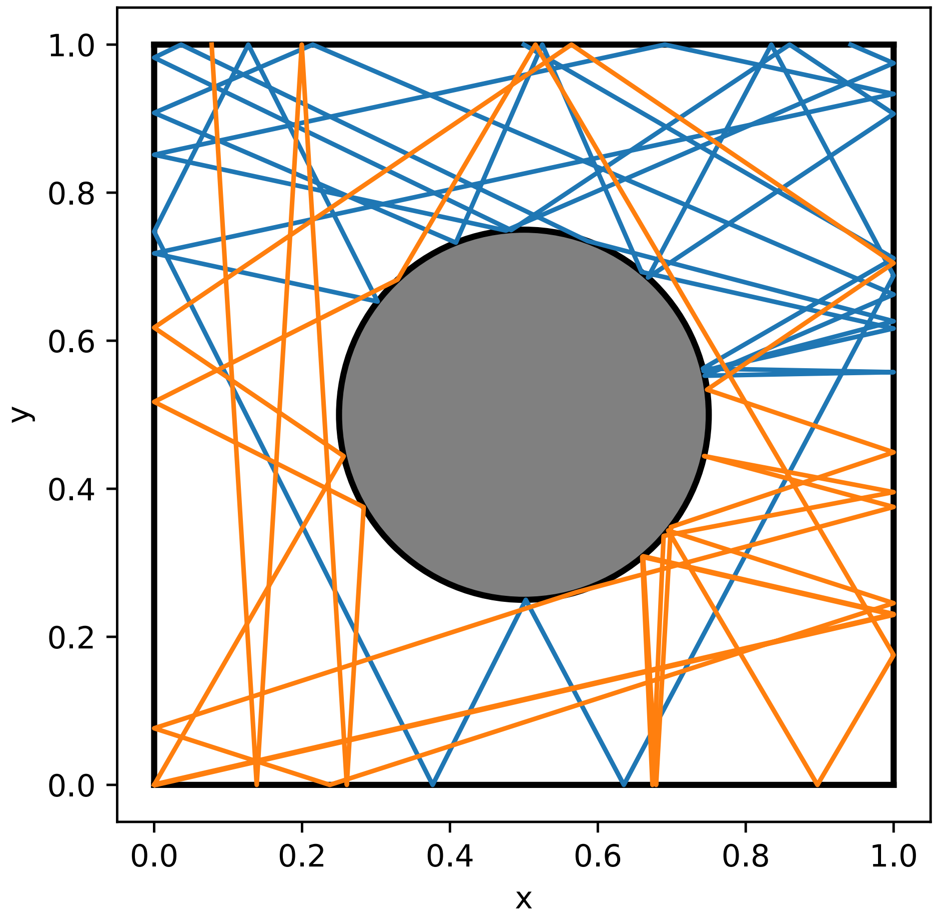
- Part of the work consisted on developing a numerical code on *Julia* to plot trajectories for given initial particle conditions.



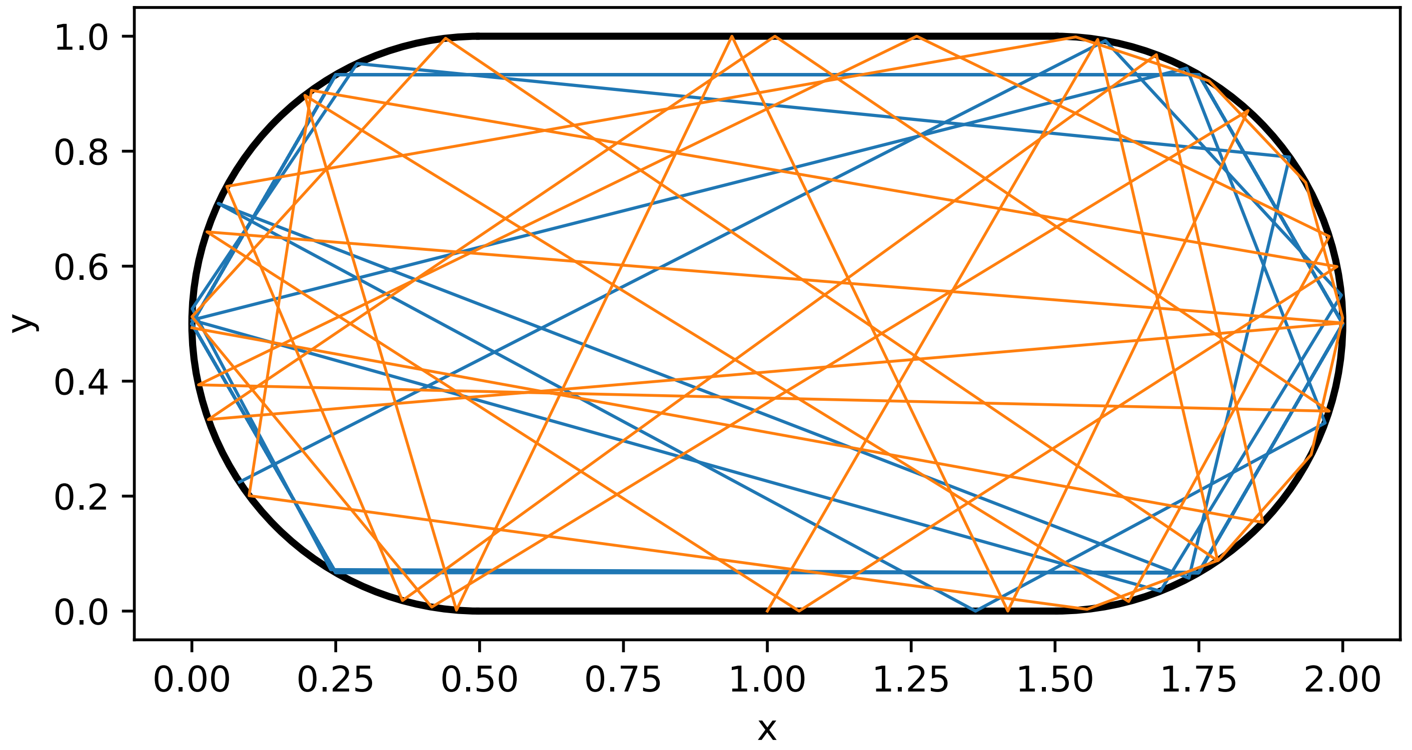
Circular billiard



Elliptical billiard



Sinai billiard



Bunimovich stadium

LYAPUNOV EXPONENT

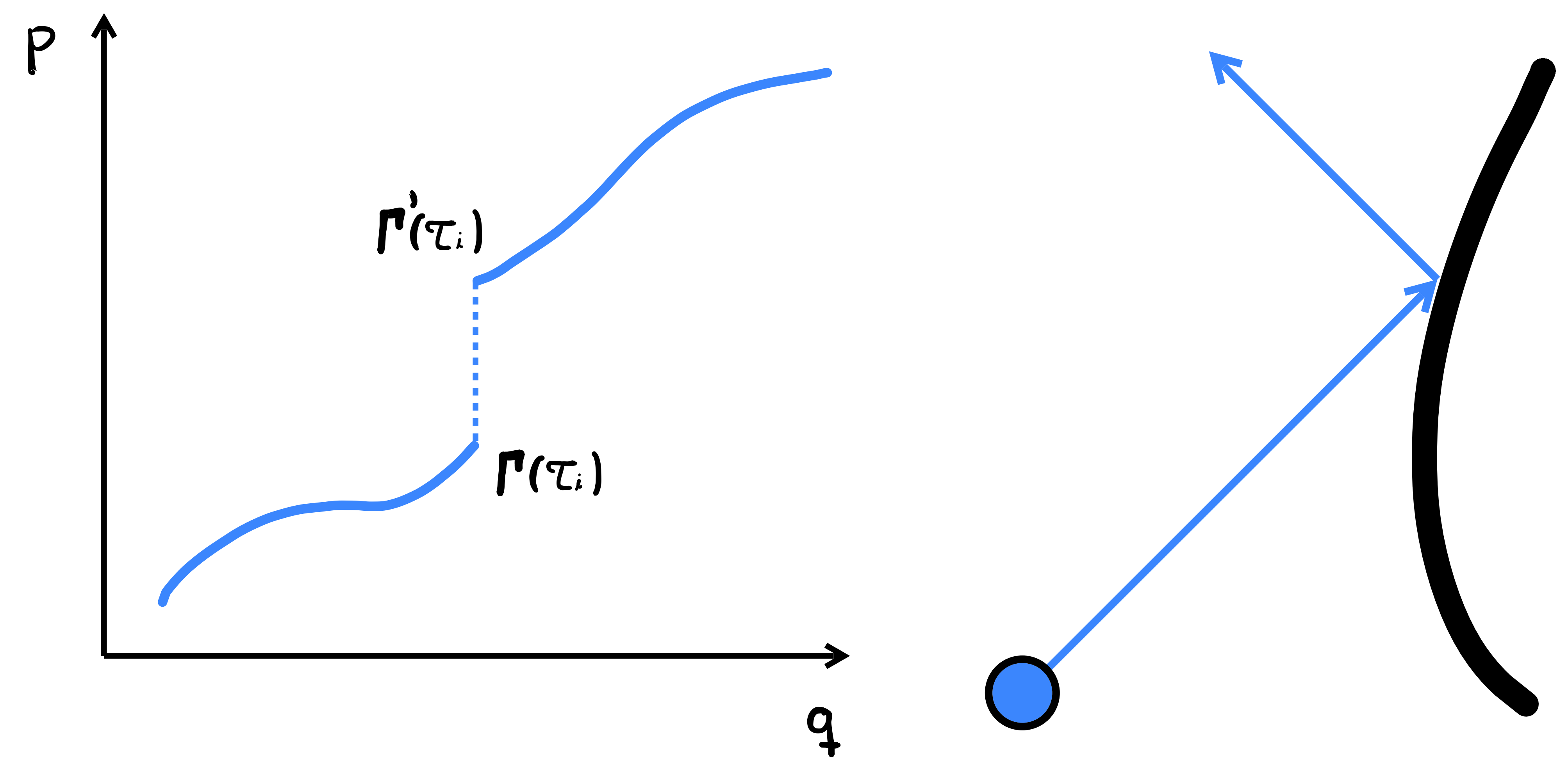
- The Lyapunov exponent (LE) gives the average exponential rate of divergence of infinitesimally nearby initial conditions.
- For a 2D billiard Lyapunov spectrum is given by: $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$.
- A positive maximal LE λ_1 calculation is enough as a chaos indicator.
  $\lambda_1 = -\lambda_4 \quad , \quad \lambda_2 = \lambda_3 = 0$

OFFSET VECTOR

- We choose a vector that measures the separation (on phase space) of two trajectories. Lets say reference $\boldsymbol{\Gamma}(t)$ and satellite $\boldsymbol{\Gamma}_s(t)$ trajectories.
- Every collision implies discontinuities on the trajectories.
- Here we are dealing with a functional composition of continuum and discrete maps.

$$\lim_{s\to 0}\boldsymbol{\Gamma}_s(t) = \boldsymbol{\Gamma}(t)$$

$$\delta\boldsymbol{\Gamma}(t) = \lim_{s\to 0}\frac{\boldsymbol{\Gamma}_s(t) - \boldsymbol{\Gamma}(t)}{s}$$



- For a point $\mathbf{\Gamma}(t_0)$ in the phase space at an initial condition $t_0 = 0$ we have time evolution given by $\mathbf{\Gamma}(t) = \Phi\left[\mathbf{\Gamma}(0)\right]$
  whereas for a discontinuity we have a map $\mathbf{\Gamma}'(t) = \mathbf{M}\left[\mathbf{\Gamma}(t)\right]$

- Whole evolution of a particle in phase space can be written down as
  $\mathbf{\Gamma}(t) = \mathbf{M} \circ \Phi^{\tau_N - \tau_{N-1}} \circ \mathbf{M} \circ \Phi^{\tau_{N-1} - \tau_{N-2}} \circ \ldots \circ \mathbf{M} \circ \Phi^{\tau_1}\left[\mathbf{\Gamma}(0)\right]$

- Following this idea the whole evolution of an initial separation of two trajectories can be expressed as
  $\delta\mathbf{\Gamma}(t) = \mathbf{S} \cdot \mathbf{L}^{\tau_N - \tau_{N-1}} \cdot \mathbf{S} \cdot \mathbf{L}^{\tau_{N-1} - \tau_{N-2}} \cdot \ldots \cdot \mathbf{S} \cdot \mathbf{L}^{\tau_1}\left[\delta\mathbf{\Gamma}(0)\right]$

CALCULATING MAXIMAL LYAPUNOV EXPONENT

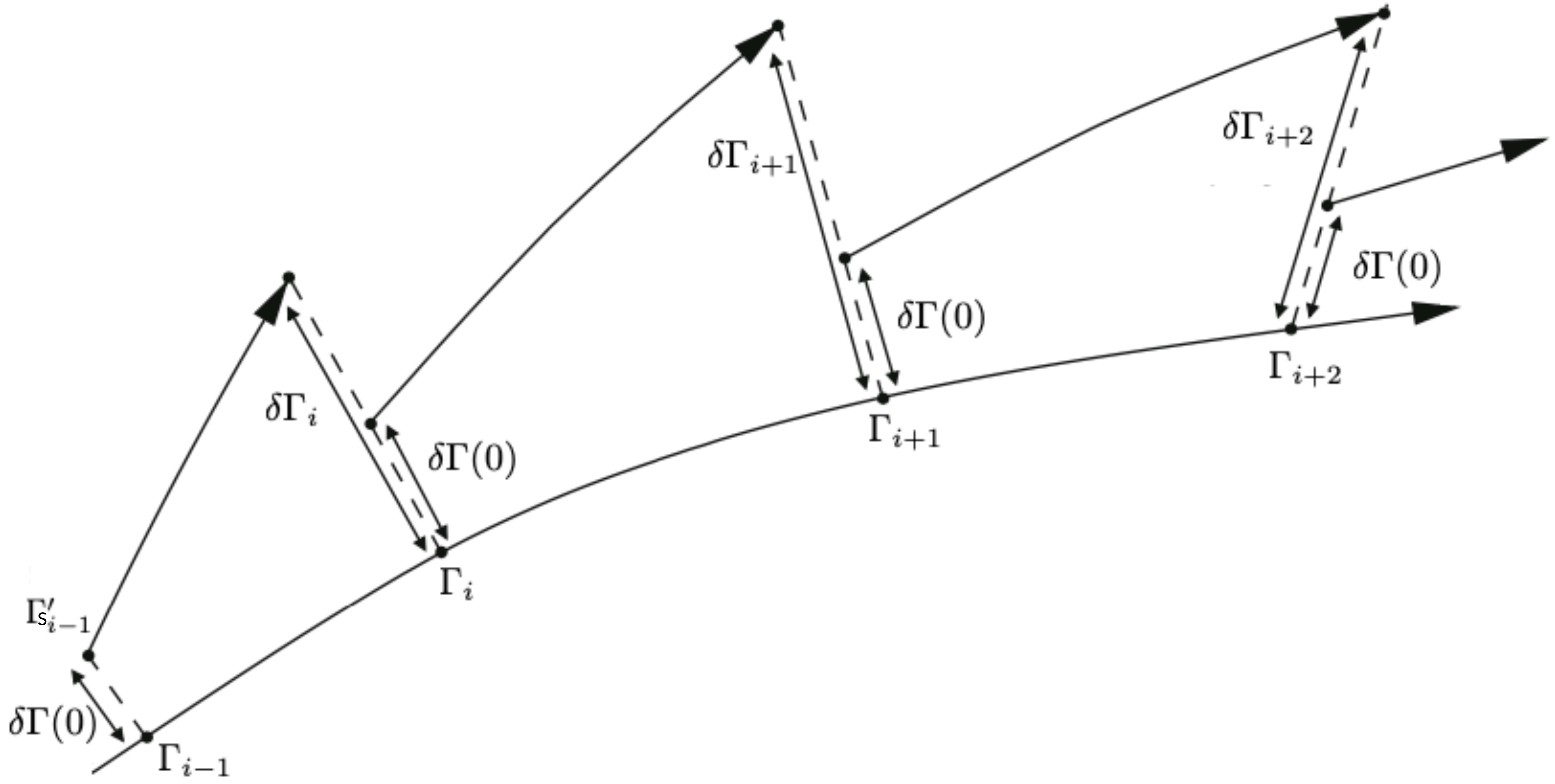
- $\lambda(\mathbf{\Gamma}_0, \delta\mathbf{\Gamma}_0) = \lim_{N\to\infty} \frac{1}{N} \ln \left| \frac{\delta\mathbf{\Gamma}_N}{\delta\mathbf{\Gamma}_0} \right|$

- LE does not depend on initial conditions so $\delta\mathbf{\Gamma}_0$ normalized to one $\lambda_i \approx \frac{1}{\Delta\tau_i} \ln \left| \delta\mathbf{\Gamma}_N \right|$, where the set of evolution times is $t = \{\tau_1, \tau_2, \ldots, \tau_N\}$.

- Except for a set of initial conditions of Lebesgue measure zero, the mean time between collisions is the same. So we can write $\bar{\lambda} = \frac{1}{N} \sum_{i=1}^{N} \frac{\ln \left| \delta\mathbf{\Gamma}(\tau_i) \right|}{\Delta\tau_i} = \frac{1}{t_N} \sum_{i=1}^{N} \ln \left| \delta\mathbf{\Gamma}(\tau_i) \right|$, where $t_N = N\Delta\tau$.

- We can expect numerical overflow because large numbers. To deal with this we again renormalize to one our offset vector for every collision.

- Now we can calculate the maximal LE knowing the continuum and discrete evolution of the offset vector.

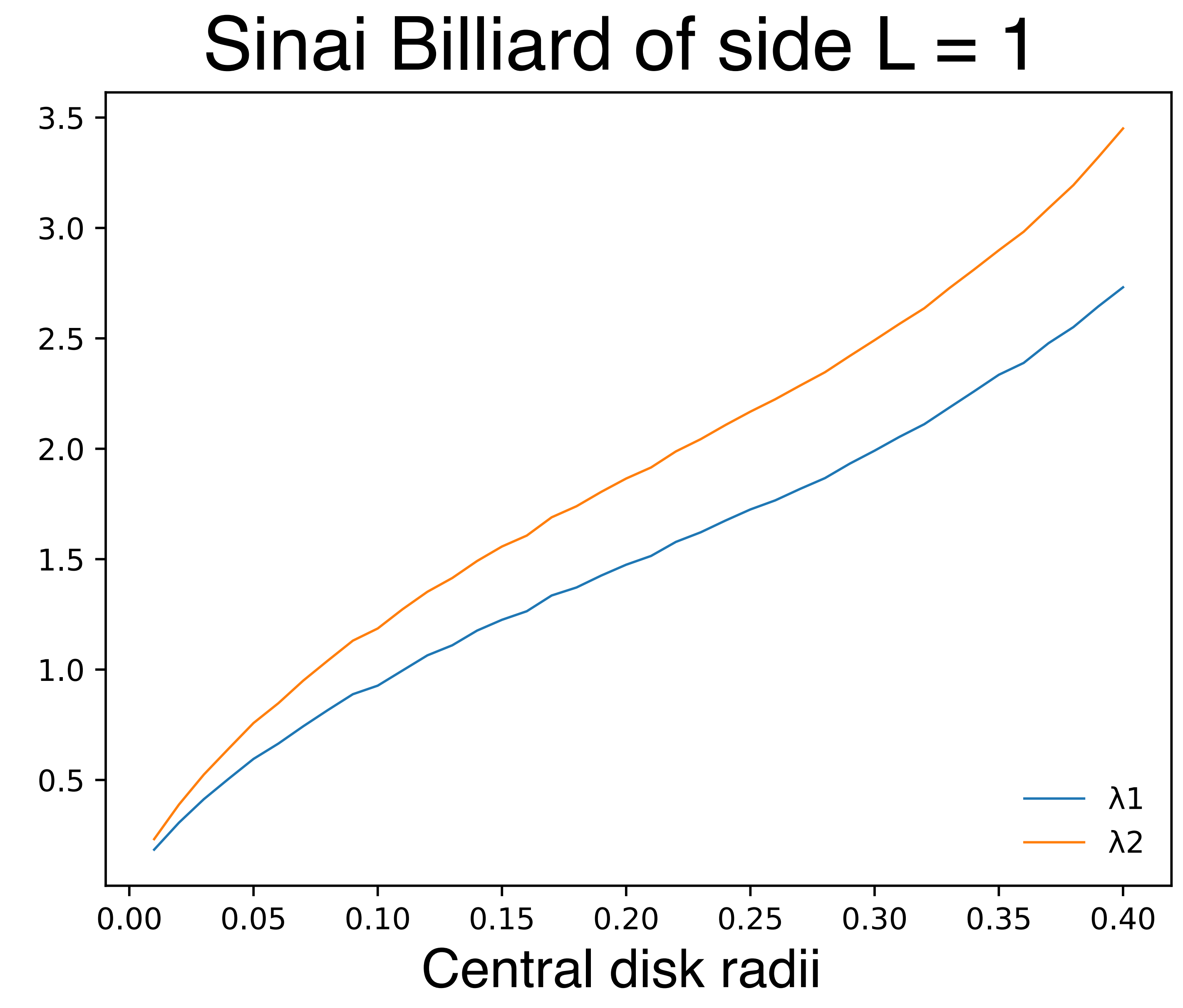


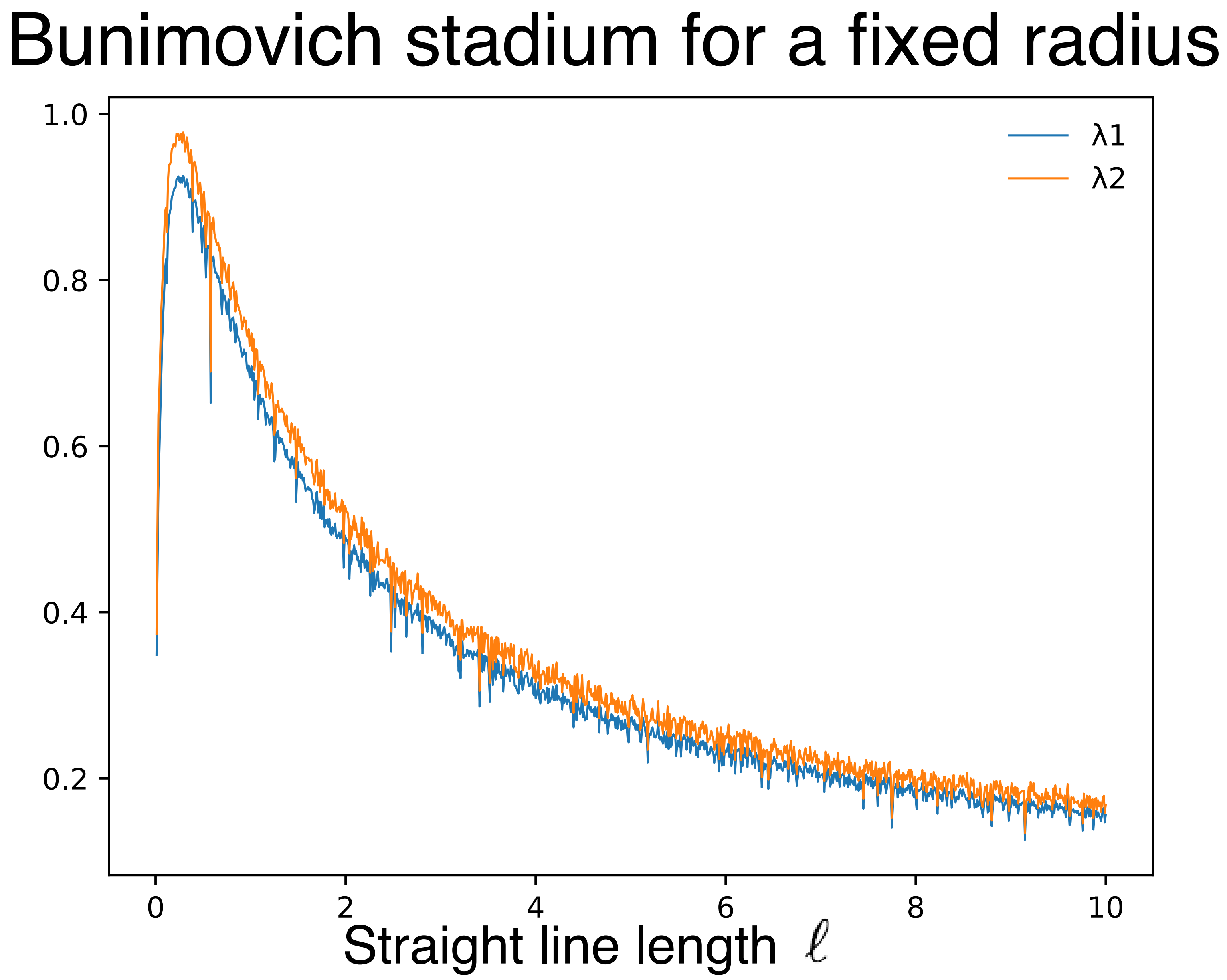
$$\mathbf{L}^{\Delta\tau_i}(\delta\boldsymbol{\Gamma}) = (\delta\mathbf{q} + \Delta\tau_i \cdot \delta\mathbf{p}\,,\delta\mathbf{p})$$

$$\delta\boldsymbol{\Gamma}' = \left[\delta\mathbf{q} - 2(\delta\mathbf{q}\cdot\mathbf{n})\mathbf{n}\,,\delta\mathbf{p} - 2(\delta\mathbf{p}\cdot\mathbf{n})\mathbf{n} - 2\gamma_r\frac{\delta\mathbf{q}\cdot\mathbf{e}}{cos(\phi)}\mathbf{e}'\right]$$

EVOLUTION MEAN TIME

- We approximated for the mean maximal LE calculation that the mean time between collisions is the same.
- $\lambda 1$ denotes the mean maximal LE using this property. Whereas $\lambda 2$ denotes the same quantity but in the calculation has been used each evolution time.





TWO WAYS FOR CALCULATING MAXIMAL LYAPUNOV EXPONENT

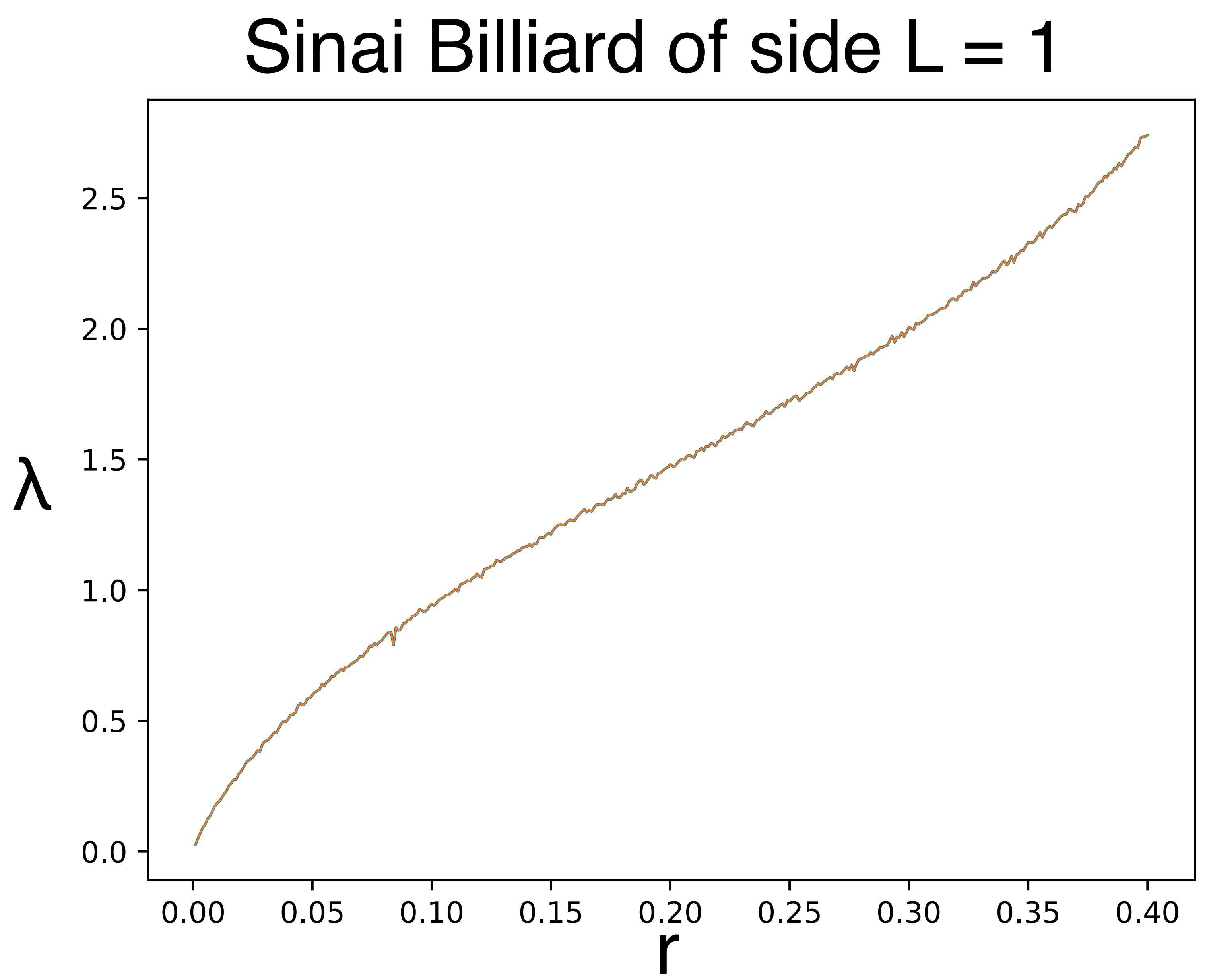
- Lets call this as the **via change ratio**:

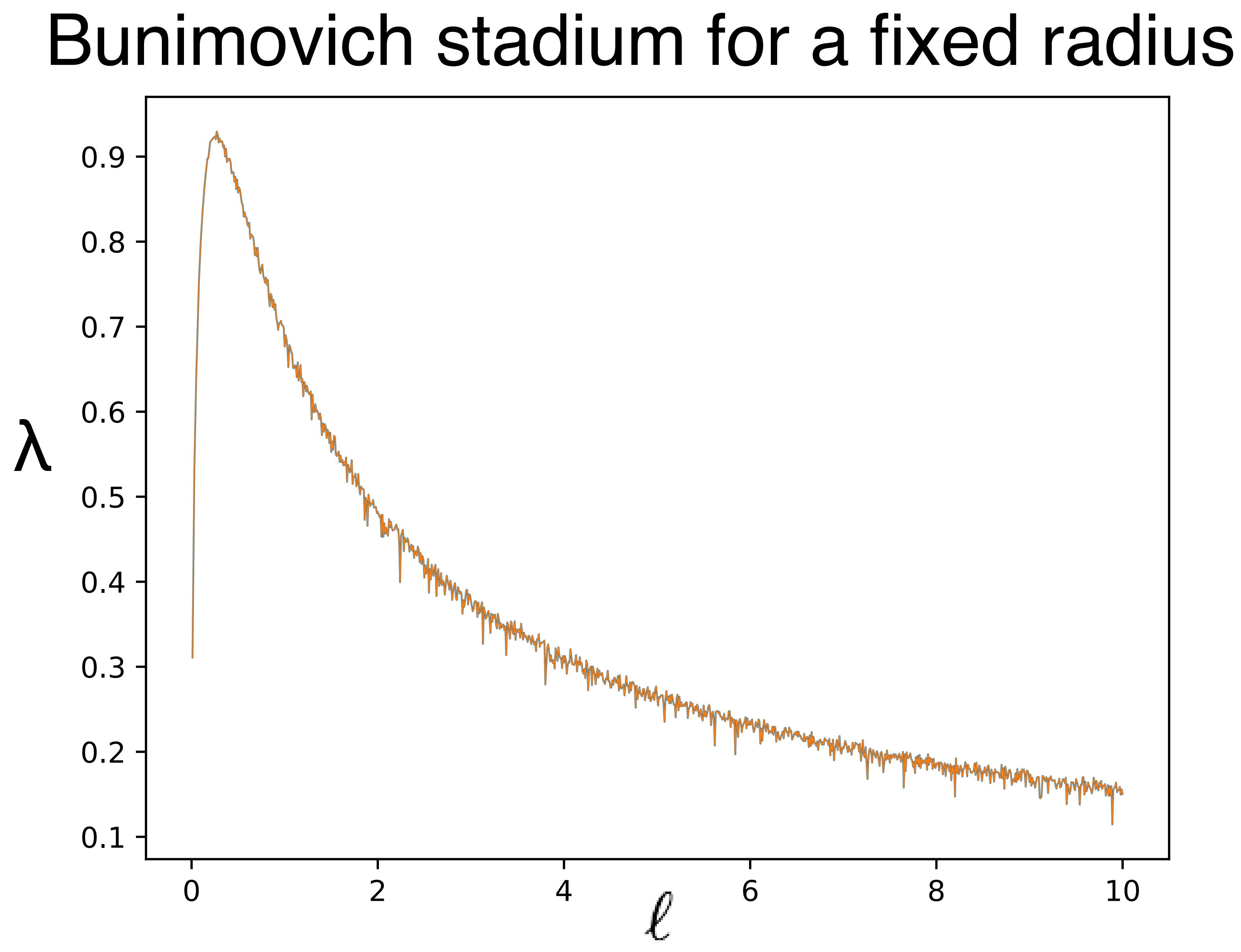
$$\bar{\lambda}_{t_N}(\mathbf{\Gamma}_0) = \frac{1}{t_N}\sum_{i=1}^{N} \ln \left| \delta\mathbf{\Gamma}(\tau_i) \right|$$

- Lets call this way as the **via eigenvalues**:

$$\bar{\lambda}_{t_N}(\mathbf{\Gamma}_0) = \frac{1}{2t_N}\sum_{i=1}^{N} \ln(\Lambda_{1,\tau_i})$$

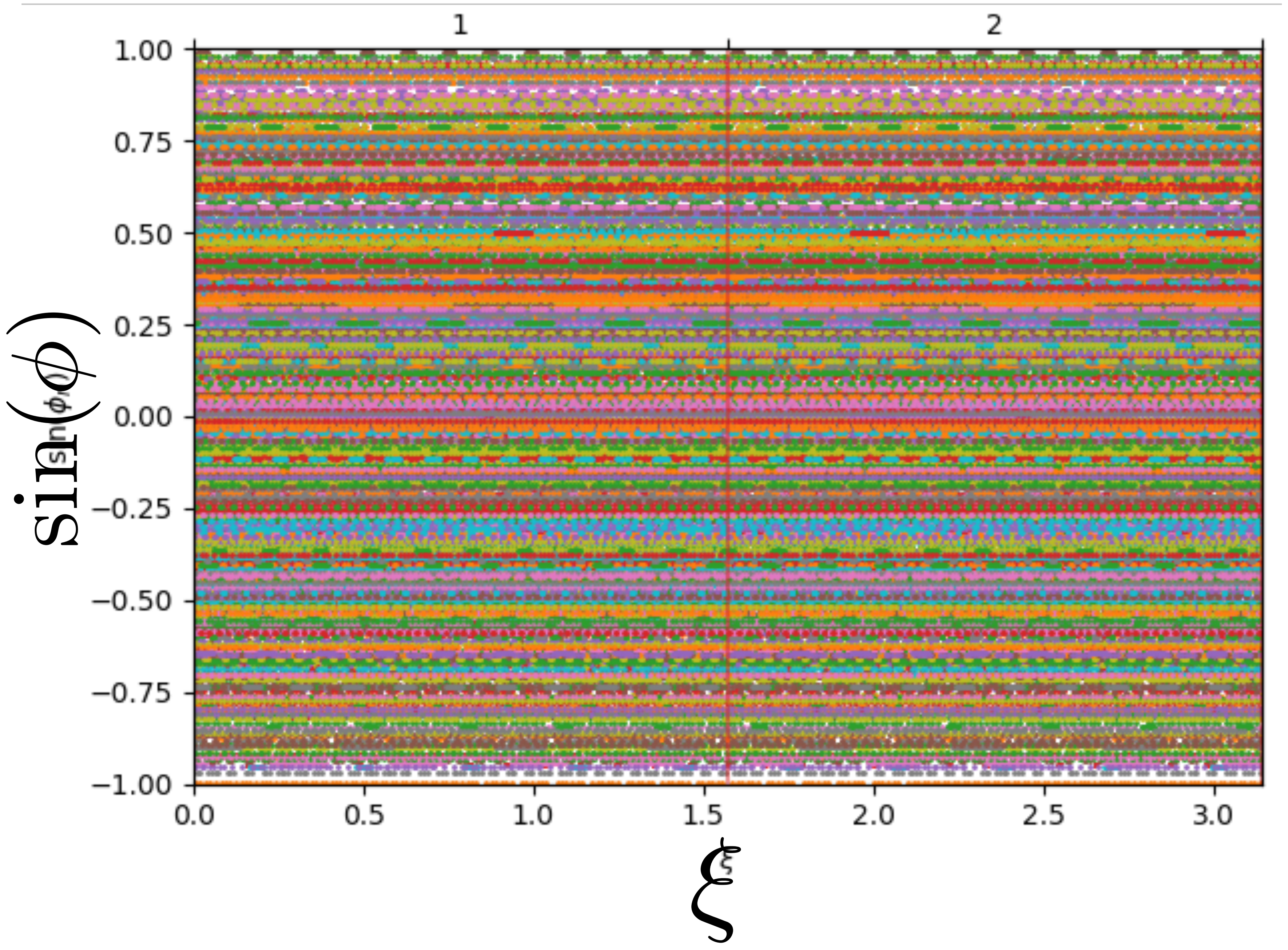
where Λ_{1,τ_i} is the major eigenvalue of $\mathbf{\Lambda}(\mathbf{\Gamma}_i) = \delta\mathbf{\Gamma}^{\dagger}_{\tau_i}\delta\mathbf{\Gamma}_{\tau_i}$.





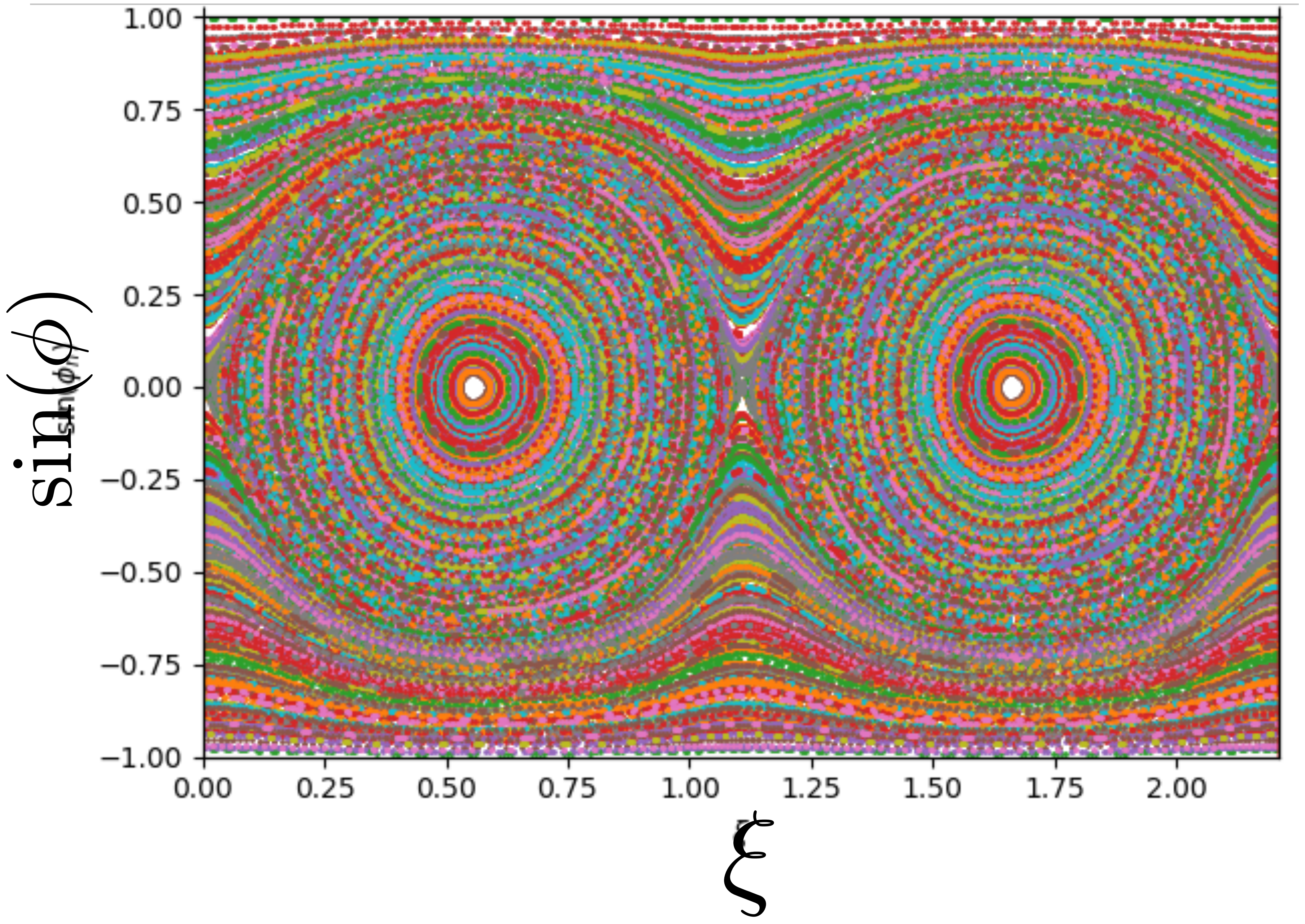
POINCARÉ–BIRKHOFF COORDINATES

- The point of impact and the direction of the trajectory can be recorded by the two values: arc length ξ and the projection of the vector velocity over the tangent to the boundary curve $\sin(\phi)$ (tangential velocity), where ϕ is the angle of incidence. These are the Poincaré-Birkhoff coordinates (PBC).
- PBC define the collision space which is nothing but a Poincaré section.



Circular billiard

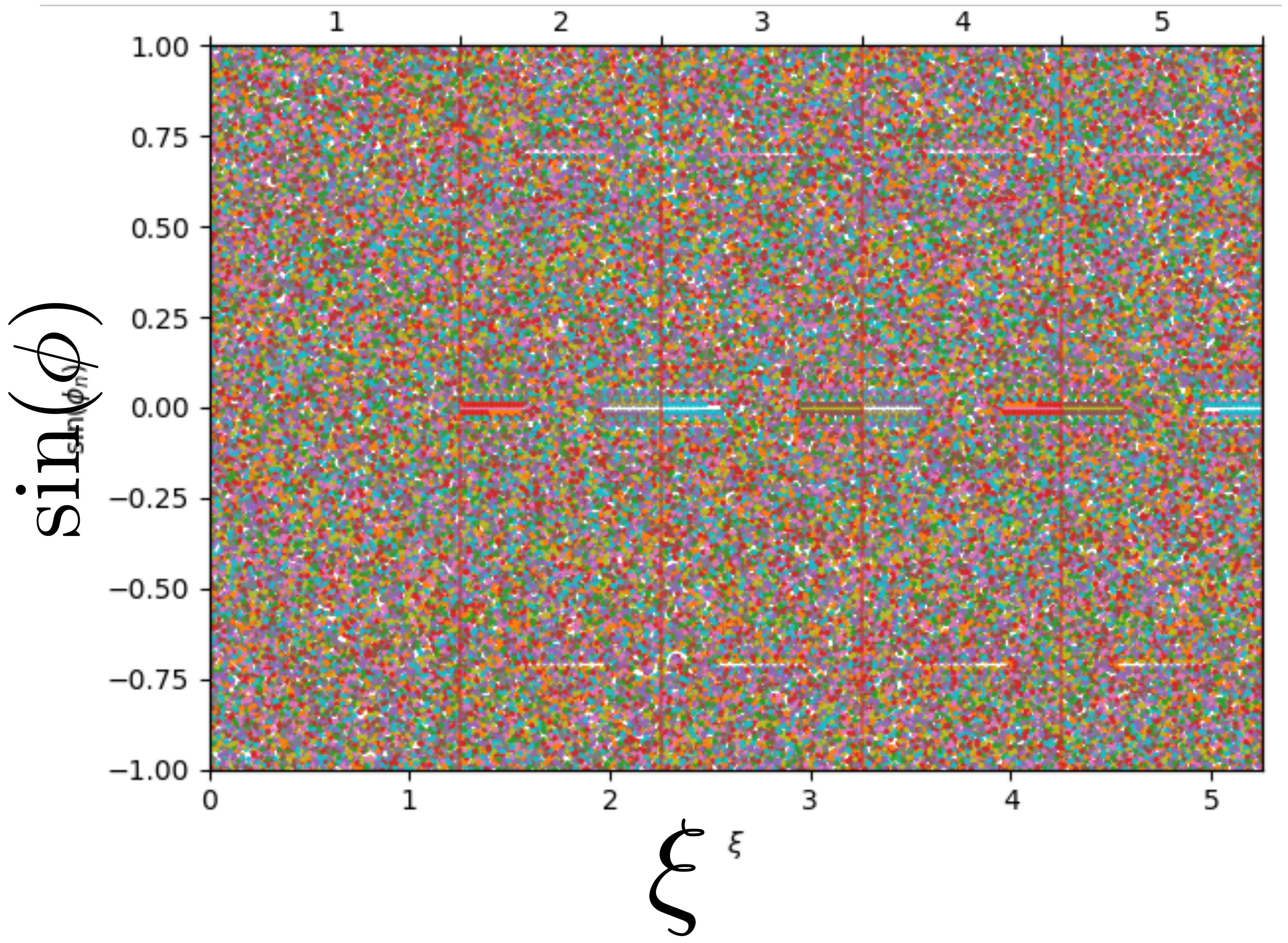
$r = 0.5$



Elliptical billiard

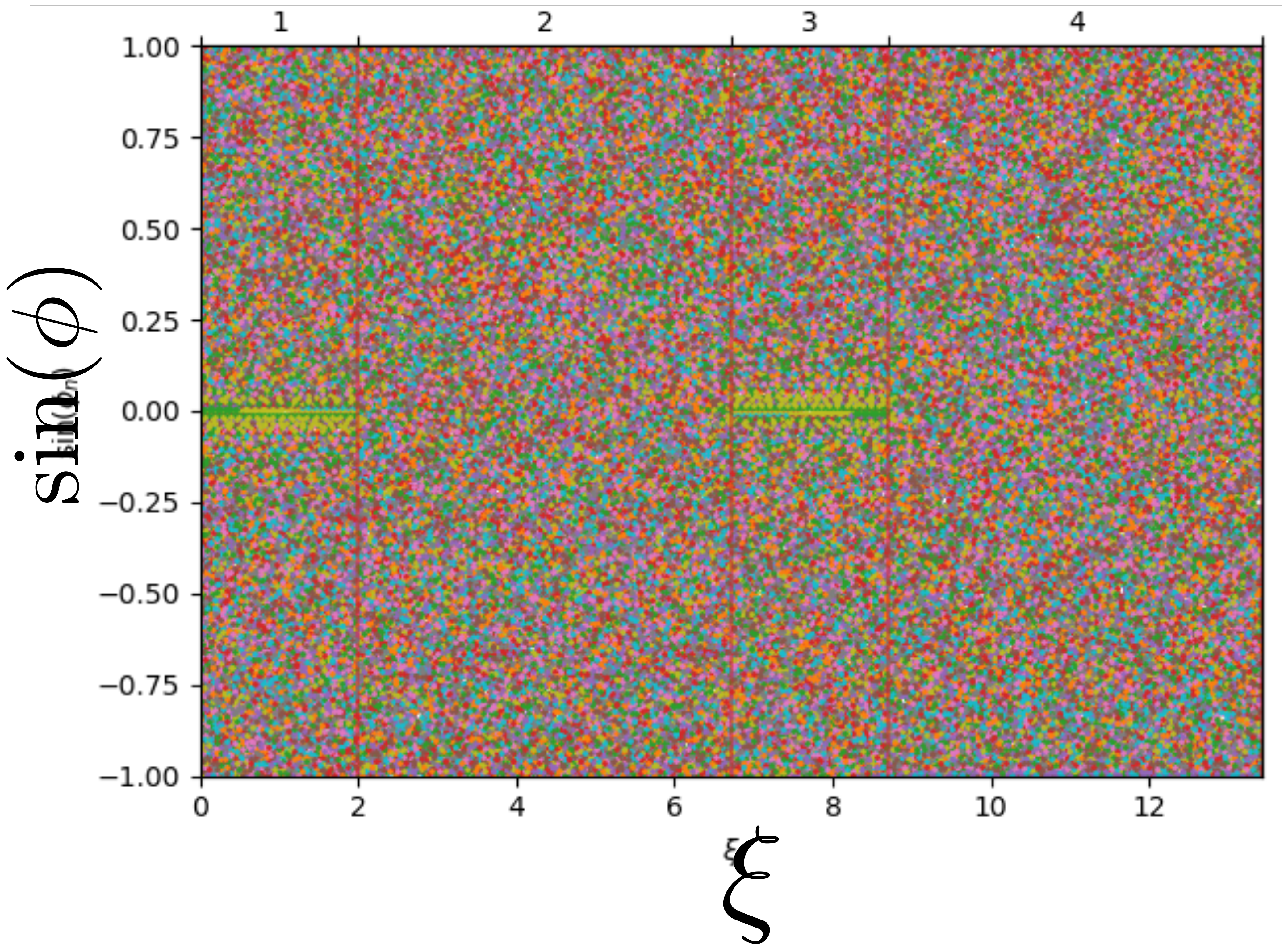
$a = 0.4 \qquad b = 0.3$

COLLISION SPACE



Sinai billiard

$L = 1 \qquad r = 0.2$

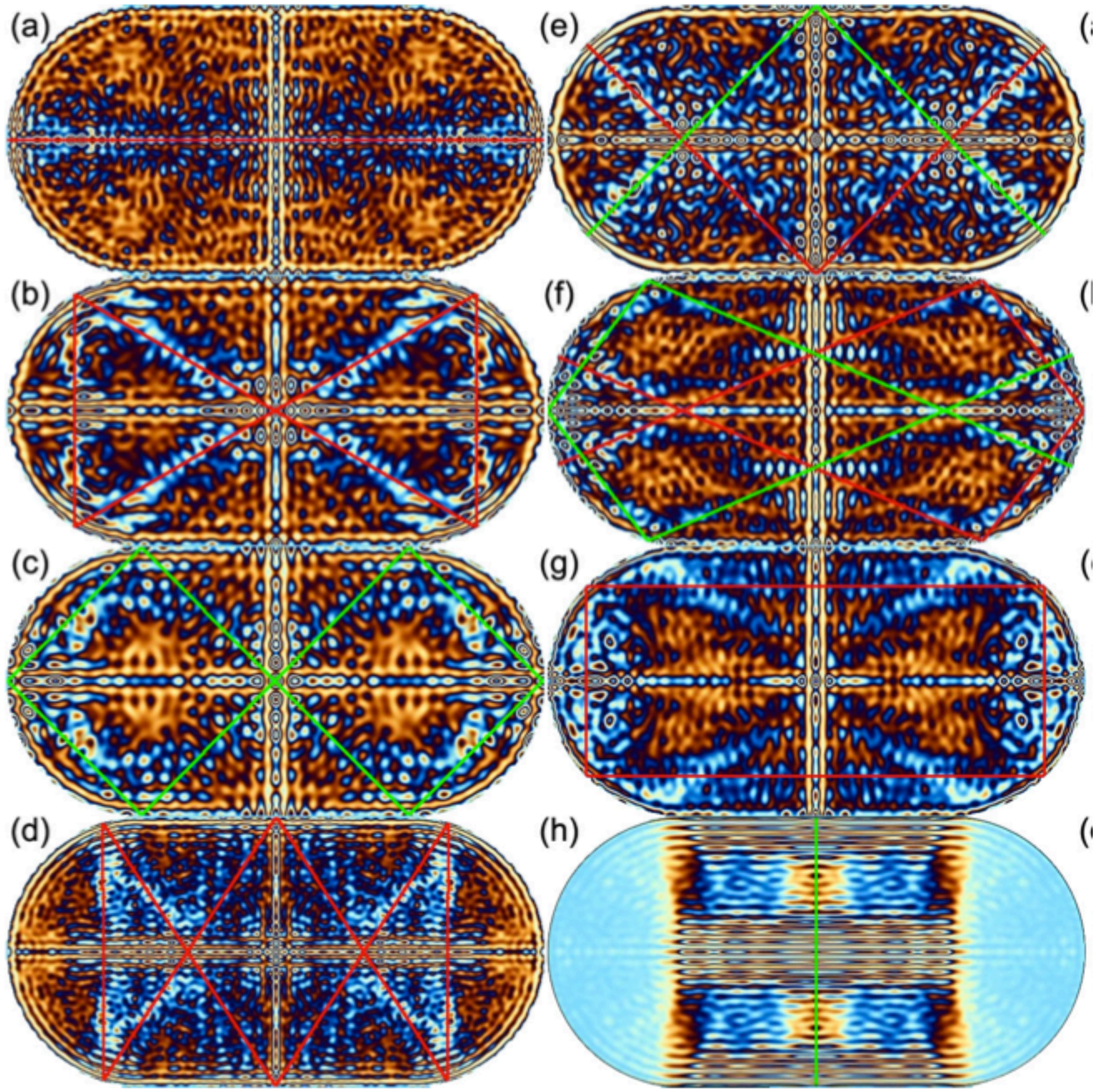


Bunimovich stadium

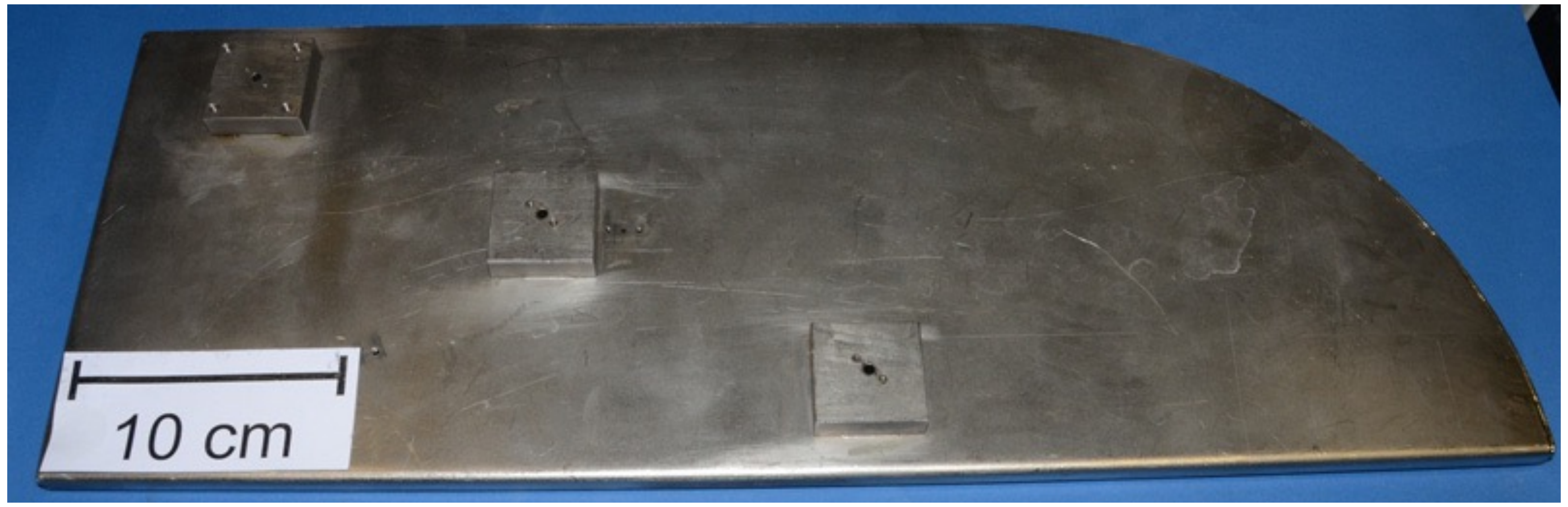
$\ell = 1 \qquad r = 1.5$

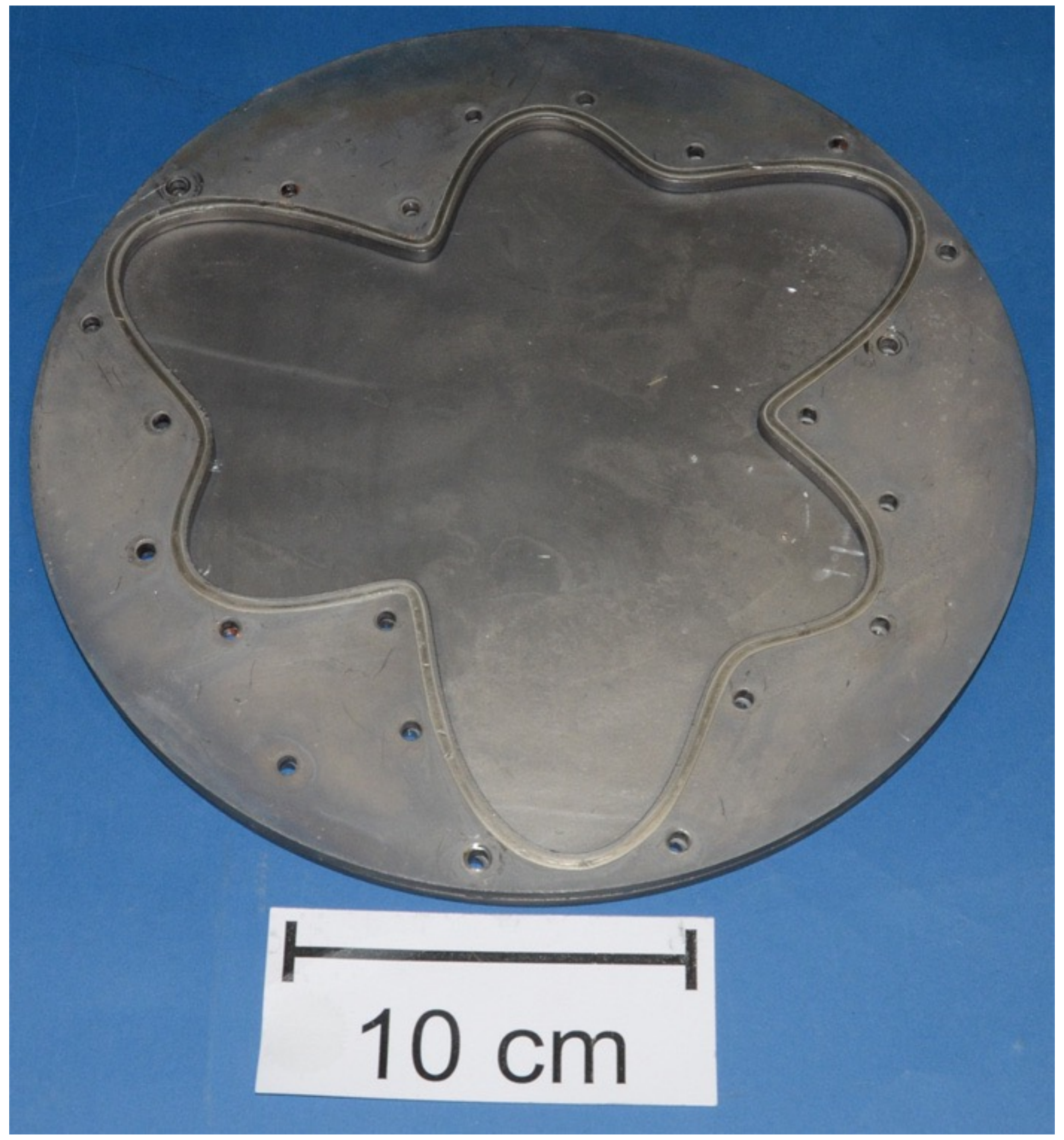
SEMICLASSICAL AND QUANTUM APPROACH FOR BILLIARDS

- For the semiclassical approach has been found evidence of scarring on billiards.
- For example, this figure is discussed on King's paper. Consists on scars formed by superimposing the squared wave function.



- Experiments were performed in the last two decades with superconducting microwave billiards.
- From the work of Dietz and Richter, we can see two experiments for 2D microwave billiards. First, it is shown a quarter of a Bunimovich stadium. Next, a random boundary.





CONCLUSIONS

- Billiards are useful as simplified descriptions of classical dynamics and chaos.
- We reviewed the Lyapunov exponent as a chaos indicator.
- We worked in how to deal with discontinuities for dynamical maps.
- We explored Poincaré-Birkhoff coordinates as a tool for chaos visualization.
- Experimental, quantum and semiclassical work on billiard systems are been carrying out recently.

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