Iniciación a la QCD

⋄ ⋄ ⋄ Segunda Sesión ⋄ ⋄ ⋄

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Sum over pol of external states: QED vs QCD

Photon emission processes

$$k_1^{\mu_1} k_2^{\mu_2} \mathcal{M}_{\mu_1 \mu_2} = 0 \quad (1)$$

and in general, n photons, $k_1^{\mu_1} k_2^{\mu_2} \cdots k_n^{\mu_n} \mathcal{M}_{\mu_1 \cdots \mu_n} = 0$ and the production of *any* number of unphysical photons vanishes.

BUT: What happens in gluon emission processes?

- In this case $k_1 \cdot \mathcal{M} \propto \epsilon_2 \cdot k_2$, which vanishes only for a *physical* ϵ_2.
- Production of one physical gluon and one unphysical gluon is equal to 0
- Production of a pair of unphysical gluons is allowed!

If $\epsilon_2 \cdot k_2 \neq 0$, then $\mathcal{M}_0$ is not equal to $\mathcal{M}_3$, and $\sum \epsilon_\mu \epsilon_\nu^* \neq -g_{\mu\nu}$.

Non-Abelian gauge invariance

D_3's gauge variation:

$$k_1 \cdot D_3 = g^2 f^{abc} \lambda^c V_0 \left[\overline{v}(\overline{q}) \not{\epsilon}_2 u(q) - \frac{k_2 \cdot \epsilon_2}{2 k_1 k_2} \overline{v}(\overline{q}) \not{k}_1 u(q) \right] \quad (2)$$

cancels the gauge var $D_1 + D_2$ for $V_0 = 1$

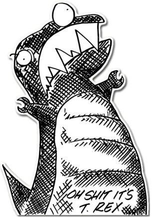
vanishes for physical gluon $k_2 \cdot \epsilon_2 = 0$

$D_1 + D_2 + D_3$ is gauge invariant (unlike QED) only for physical external on-shell gluons.

Sum over physical and unphysical external states

Homework: show that

$$\sum_{non-physical} |\epsilon_1^{\mu}\epsilon_2^{\nu}\mathcal{M}_{\mu\nu}|^2 = \left|ig^2 f^{abc}\lambda^c \frac{1}{2k_1 k_2}\overline{v}(\overline{q})\not{k}_1 u(q)\right|^2 \qquad (3)$$



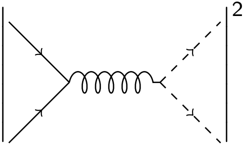
Physical vs unphysical external states in QCD

In QCD we need to restrict this sum over polarizations

- unitarity: off-shell propagators to physical dof $\rightsquigarrow$ physical gauges
- or study implications of gauge-fixing in non-physical gauges for quantization procedure
  - appearance of "*ghosts*": two colour-octet scalar dof
  - ghost rôle enforce unitarity in non-physical gauges
  - they will appear in internal closed loops or pair-produced in final states
  - they only couple to gluons
  - each closed loop comes with -1 sing, as if they obeyed Fermi statistics
- scalars: prescription breaks spin-statistics relation $\rightsquigarrow$ production probabilities could be negative
- what we needed to cancel contributions of non-transverse dof appearing in non-physical gauges

Physical vs unphysical external states in QCD

Adding the ghost contribution to $q\bar{q} \longrightarrow gg$ decays gives



$$\Big|\cdots\Big|^2 = -\left| ig^2 f^{abc}\lambda^c \frac{q}{2k_1 k_2}\bar{v}(\bar{q})\not{k}_1 u(q)\right|^2 \qquad (4)$$

which exactly cancels the contribution of non-transverse gluons in the non-physical gauge in eq. (3).

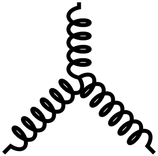
See in textbooks detailed derivation of the need for ghosts and properties.
Throughout the rest of the lectures use physical gauges.

Gauge invariance and Feynman rules

$$= \quad \delta^{ab} \frac{-ig^{\alpha\beta}}{p^2 + i\epsilon}$$

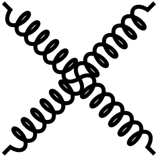
$$= \quad \delta^{ab} \frac{i}{p^2 + i\epsilon}$$

$$= \quad \delta^{ik} \left. \frac{i}{p^2 - m + i\epsilon} \right|_{mn}$$



$$= \quad gf^{abc} \left[g^{\alpha\beta}(p-q)^\gamma + g^{\beta\gamma}(q-r)^\alpha + g^{\gamma\alpha}(r-p)^\beta \right]$$

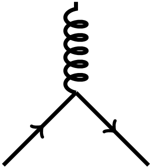
Gauge invariance and Feynman rules



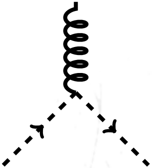
$$= \; -ig^2 f^{xac} f^{xbd} \left(g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\beta\gamma} \right)$$

$$- \; ig^2 f^{xad} f^{xbc} \left(g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta} \right)$$

$$- \; ig^2 f^{xab} f^{xcd} \left(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma} \right)$$



$$= \; ig \lambda^a_{ki} \gamma^\alpha_{mn}$$



$$= \; -gf^{abc} q^\alpha$$

3 lectures: *three quarks for Muster Mark!*

Lecture 1:

- QCD at the LHC
- Gauge invariance and Feynman rules for QCD

Lecture 2:

- Renormalization and running α_s
- pQCD in e^+e^--collisions: from partons to hadrons, jets, shape variables

Lecture 3:

- pQCD in lepton-hadron collisions: DIS and parton evolution
- pQCD at the LHC

√ Overlap with other lectures is unavoidable.

√ As with other things in life, the U and everything:

overlap is good :)

Colour algebra technology

Since there are lots of λ^a_{ij} and f^{abc}, useful to have some technology at hand. Assume gauge group is $SU(N)$ so

$$[\lambda^a, \lambda^b] = if^{abc}\lambda^c \tag{5}$$

with f^{abc} totally antisymmetric $\rightsquigarrow$ λ matrices are traceless. Noteworthy

- we always sum over initial/final/intermediate state colours, so no need for explicit values of f^{abc}
- all the results can be expressed using group invariants *Casimirs*

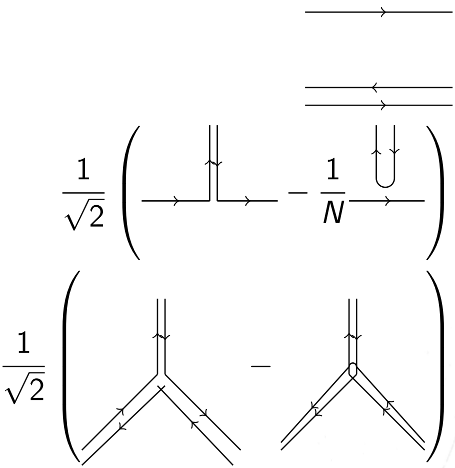
T_F fixes λ normalization

$$tr(\lambda^a\lambda^b) = T_F\delta_{ab} \tag{6}$$

where convention $T_F = 1/2$ for fund rep

Colour flows!: color algebra graphical tricks

Use rep for q/g propagators and $q\bar{q}g/ggg$ vertices



Quark (7)

Gluon (8)

QGV (t^a) (9)

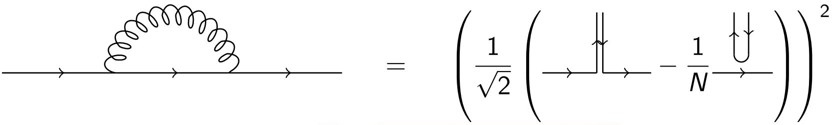
TGV (f^{abc}) (10)

Colour algebra graphical tricks

- colour index contraction $\rightsquigarrow$ connect the respective colour (or anticolour) lines
- closed loop of a colour line $\equiv$ trace of the unit matrix $\rightsquigarrow$ N factor
- rep of $q\bar{q}g$ vertex embodies the idea of "colour conservation"
- colour-anticolour quantum numbers carried by $q\bar{q}$ pair transferred to the gluon
- piece $\propto 1/N$ in $q\bar{q}g$ vertex only when colour of q and $\bar{q}$ are the same. Why?

Colour algebra graphical tricks

As a first example of applications, let us evaluate C_F :



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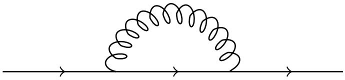
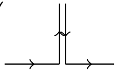
Colour algebra graphical tricks

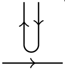
As a first example of applications, let us reevaluate C_F :



$$= \frac{1}{\sqrt{2}} \Bigg($$



$$- \frac{1}{N}$$



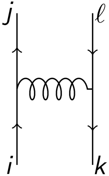
$$\Bigg)^2$$

$$= \delta^{ij} \frac{N^2 - 1}{2N} \qquad (12)$$

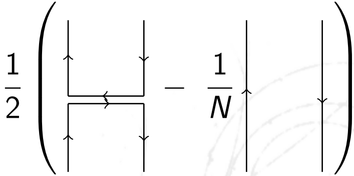
Colour algebra graphical tricks

Homework: calculate the colour factor for $q\bar{q} \longrightarrow q\bar{q}$ scattering, and show that:

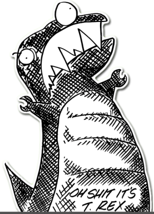
$$\sum_a (\lambda^a)_{ij}(\lambda^a)_{\ell k} \quad =$$



$$=$$

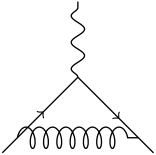


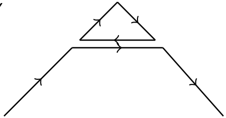
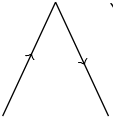
$$= \frac{1}{2}\left(\delta_{ik}\delta_{\ell j} - \frac{1}{N}\delta_{ij}\delta_{\ell k}\right) \qquad (13)$$



Colour algebra graphical tricks

This result can be used to evaluate the one-loop colour factors for the interaction vertex with a photon:

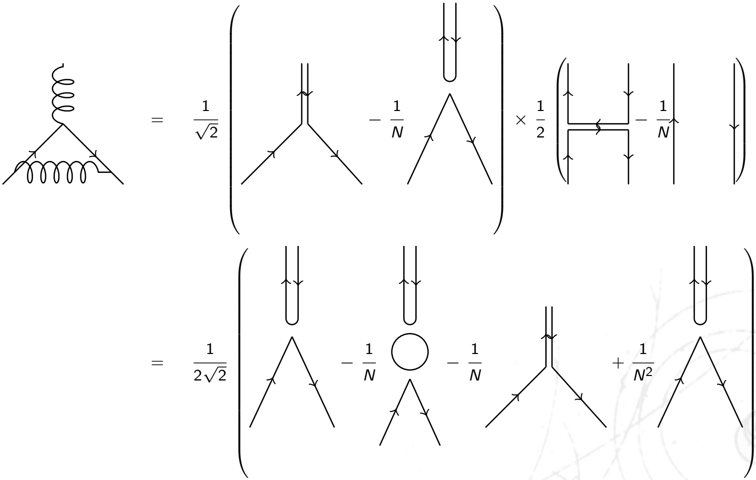


$= \frac{1}{2}\Bigg($  $- \frac{1}{N}$  $\Bigg)$

$$= \frac{1}{2}\frac{N^2-1}{N}\delta_{ij} = C_F\delta_{ij} \qquad (14)$$

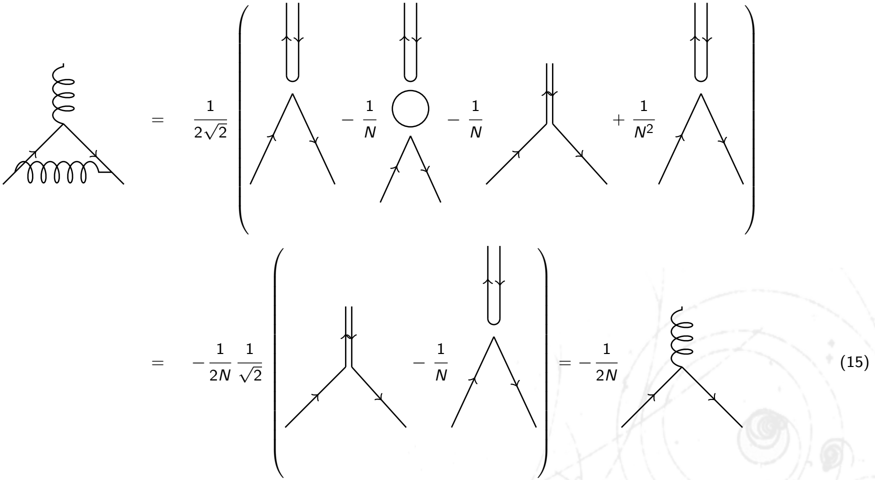
Colour algebra graphical tricks

For the interaction with a gluon we have instead:



Colour algebra graphical tricks

Contd.



Colour algebra

Finally,

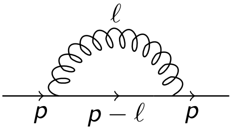
- note that in the case of the coupling to the photon the $q\bar{q}$ pair is in colour-singlet state
- the gluon exchange effect in this case has a positive sign ($\Rightarrow$ attraction)
- the case of the coupling to the gluon the $q\bar{q}$ pair is in colour-octet state, and the gluon-exchange correction has a negative sign relative to the Born interaction
- the force between a $q\bar{q}$ pair is therefore *attractive* if the pair is in a colour-singlet, while it is *repulsive* if it is in a colour-octet state!

Renormalization

- QCD calculations are extremely demanding
- Although perturbative, the size of the coupling constant even at rather large values of exchanged momentum, Q^2 is such that the convergence of the perturbative expansion is slow
- Several orders of perturbation theory (PT) are required in order to obtain a good accuracy
- The complexity of the calculations grows dramatically with the order of approximation
- The evaluation of a large class of higher-order diagrams gives rise to results which are a priori ill-defined, namely to infinities

Renormalization: UV divergence

Appears when considering the corrections to the quark self-energy:



$$= (-ig)^2 C_F \int \frac{d^4\ell}{(2\pi)^4} \gamma_\mu \frac{i}{\not p + \not \ell} \gamma_\nu \left(-\frac{ig^{\mu\nu}}{\ell^2}\right)$$

$$\equiv i\not p \Sigma(p) \qquad (16)$$

where

$$\Sigma(p) = iC_F \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 (p+\ell)^2}, \qquad (17)$$

- log divergent in the UV ($|\ell| \longrightarrow \infty$) region
- why? what does it mean?
- how to deal with them?

Infinite line of charge: V and $\vec{E}$ set up

From EM lectures recall:

Wire of infinite length, carrying a constant charge density λ, λ dimensions are $[\mathrm{length}]^{-1}$. Evaluate the electric potential V and the electric field $\vec{E}$, in a point P at distance R from the wire.

$$V(R) = \int \frac{\lambda(r)}{r} dx = \lambda \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{R^2 + x^2}} \tag{18}$$

log divergent: infinity!
BUT $\vec{E}$ field is proportional to gradient of the scalar potential

$$V'(R) \sim \lambda \int_{-\infty}^{+\infty} \frac{dx}{(R^2 + x^2)^{3/2}} \tag{19}$$

convergent

Infinite line of charge: regularization

Easy to operate, introduce regularized $V(R)$ using $[\Lambda] = [L]$:

$$V_\Lambda(R) = \int_{-\Lambda}^{\Lambda} \lambda \frac{dx}{\sqrt{R^2 + x^2}} = \lambda \log \left[\frac{\sqrt{\Lambda^2 + R^2} + \Lambda}{\sqrt{\Lambda^2 + R^2} - \Lambda} \right] \tag{20}$$

So electric field

$$\vec{E}(R) = \lim_{\Lambda \longrightarrow \infty} [-\vec{\nabla} V_\Lambda(R)] \tag{21}$$

Prescription leads to expected result

$$\vec{E}(R) = \lim_{\Lambda \longrightarrow \infty} \hat{R} \frac{2\lambda}{R} \frac{\Lambda}{\sqrt{\Lambda^2 + R^2}} \rightarrow \frac{2\lambda}{R} \hat{R} \tag{22}$$

Physical observable does not depend on Λ!

Infinite line of charge: renormalization

Potential differences are also well defined even in the $\Lambda \longrightarrow \infty$ limit

$$\delta V = \lim_{\Lambda \longrightarrow \infty} [V_\Lambda(r_2) - V_\Lambda(r_1)] = \lambda \log\left(\frac{r_1^2}{r_2^2}\right) \tag{23}$$

We can *renormalize* the potential by subtracting $V(R)$ at some fixed value of $R = R_0$ and taking the $\Lambda \longrightarrow \infty$ limit:

$$V(R) \longrightarrow V(R) - V(R_0) = \lambda \log\left(\frac{R_0^2}{R^2}\right) \tag{24}$$

The non-physical infinities present in $V(R)$ and $V(R_0)$ cancel each other, leaving a finite result, with a non-trivial R-dependence

Dealing with infinities in your life: a short strategy

i) Identify an appropriate way to *regularize* divergent integrals.

ii) Absorb the divergent terms into a redefinition of fields or parameters (via e.g. subtractions) i.e. *renormalize* them.

iii) Check for *consistency*: physical results do not depend on the regularization prescription.

Fix all, cure all?

Sounds like this regularization/renormalization procedure can always be carried out, with all posible infinities being controlled/removed. This is not true!
In fact it is highly non-trivial that these procedures can be performed consistently for any possible type of divergence which develops in PT

pQCD using dimensional regularization

Typical expression (with $L = \ell - xp$ and $M^2 = x(1-x)p^2$)

$$I(M^2) = \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 + M^2]^2} \tag{25}$$

In fact quark self-energy $\Sigma(p)$ contains

$$\frac{1}{\ell^2} \frac{1}{(\ell - p)^2} = \int_0^1 dx \frac{1}{(L^2 + M^2)^2} \tag{26}$$

Different strategies

- regularize with mom cut-off and renormalize with substraction $I(M^2)$-$I(M_0^2)$
- $\surd$ experience: regularize $I(M^2)$ with analytic continuation of integral in number of space-time dims and renormalize with counterterms

pQCD using dimensional regularization

DR prescription

- D-dim integral is finite $\forall D < 4$

$$I_D(M^2) = \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(\ell^2 + M^2)^2} \tag{27}$$

- formal meaning to $I_D(M^2)$ for *continuous* values of D away from $D = 4$
- Perform all our manipulations in $D \neq 4$, regulate the divergences, renormalize fields/couplings, and then go back to $D = 4$.

We define (for Euclidean metrics):

$$d^D\ell = d\Omega_{D-1}\ \ell^{D-1} d\ell \tag{28}$$

Homework: Get it by using the following identity:

$$\int d^D\ell\ e^{-\vec{\ell}^2} \equiv \left[\int d\ell\ e^{-\ell^2}\right]^D = \pi^{D/2} \tag{29}$$

pQCD using dimensional regularization

Using this technique, you can check that

$$
\begin{aligned}
I_D(M^2) &= \frac{1}{(4\pi)^{D/2}} \frac{1}{\Gamma(D/2)} \int_0^\infty dx \; x^{\frac{D-2}{2}} (x + M^2)^2 \\
&= \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(2 - D/2)}{\Gamma(2)} (M^2)^{\frac{D}{2}-2} \qquad (30)
\end{aligned}
$$

If $D = 4 - 2\epsilon$ $(\epsilon \to 0)$ and small-ϵ expansion:

$$
\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_\epsilon + \mathcal{O}(\epsilon) \qquad (31)
$$

pQCD using dimensional regularization

We finally obtain:

$$(4\pi)^2 I_D(M^2) \longrightarrow \frac{1}{\epsilon} - \log(4\pi M^2) - \gamma_\epsilon \qquad (32)$$

infinity regularized as a pole in $(D-4)$
M-dependent is log, integral was dimensionless in $D=4$
$1/\epsilon$ pole can be removed by a subtraction:

$$I(M^2) = I(\mu^2) + \frac{1}{(4\pi)^2} \log\left(\frac{\mu^2}{M^2}\right) \qquad (33)$$

where the subtraction scale μ^2 is usually referred to as the "renormalization scale".

pQCD and renormalization

Use self-energy Eq. (16) to add a counterterm to the Lagrangian:

$$\mathcal{L} \longrightarrow \mathcal{L} + \Sigma(p)\overline{\psi} i \partial\!\!\!/ \psi = [1 + \Sigma(p)]\overline{\psi} i \partial\!\!\!/ \psi + \cdots \tag{34}$$

Corrections $\mathcal{O}(g^2)$ to inverse propagator are finite:

 + 

$$= -i p\!\!\!/ \Sigma(p) + i p\!\!\!/ \Sigma(p) = 0 \tag{35}$$

Inclusion of counterterm: interpreted as *renormalization of quark wave function*. Homework: Define

$$\psi_R = \left(1 + \Sigma(p^2)\right)^{1/2} \psi \tag{36}$$

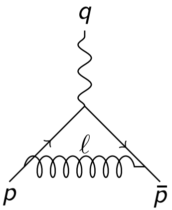
and verify that the kinetic part of the Lagrangian written in terms of ψ_R takes again the canonical form.

Renormalization in PT

Non-trivial in pQCD

Sounds like this regularization/renormalization procedure can always be carried out, with all posible infinities being controlled/removed. This is not true. In fact it is highly non-trivial that these procedures can be performed consistently for any possible type of divergence which develops in PT.

QCD @ one-loop to quark-photon intn



$$= (-ig)^2 C_F \int \frac{d^4\ell}{(2\pi)^4} \left[\gamma^\rho \frac{i}{\not p + \not \ell} \overbrace{(-ie\gamma^\mu)}^{\Gamma^\mu} \frac{i}{\not{\bar p} + \not \ell} \gamma^\rho \right] \left(\frac{-i}{\ell^2} \right)$$

$$= -ig^2 C_F \int \frac{d^4\ell}{(2\pi)^4} (-2)(\not{\bar p} + \not\ell)\Gamma^\mu(\not p + \not\ell) \frac{1}{\ell^2 (p+\ell)^2 (\bar p + \ell)^2}$$

$$= -ig^2(-2) C_F \int \frac{d^4\ell}{(2\pi)^4} \frac{\not\ell \Gamma^\mu \not\ell}{\ell^2 (p+\ell)^2 (\bar p + \ell)^2} = ie\gamma^\mu V(q^2)$$

$V(q^2)$ div removed: add counterterm to the bare Lagrangian:

$$\mathcal{L}_{int} = -eA_\mu \overline{\psi}\gamma^\mu\psi \longrightarrow \; - eA_\mu \psi\gamma^\mu\psi - eV(q^2)A_\mu\overline{\psi}\gamma^\mu\psi$$
$$= -[1 + V(q^2)]e\; A_\mu \overline{\psi}\gamma^\mu\psi \qquad (37)$$

Renormalization in PT

Taking into account both counterterms:

$$\mathcal{L}_{q,\gamma} \;=\; \left[1+\Sigma(p^2)\right]\overline{\psi}i\partial\!\!\!/\psi - [1+V(q^2)]e\,A_\mu\overline{\psi}\gamma^\mu\psi \qquad (38)$$

Defining a renormalized charge by:

$$e_R = e\,\frac{1+V(p^2)}{1+\Sigma(q^2)} \qquad (39)$$

we are left with the renormalized Lagrangian:

$$\mathcal{L}_R \;=\; \overline{\psi}_R\, i\,\partial\!\!\!/\,\psi_R \;+\; e_R A_\mu\overline{\psi}_R\gamma^\mu\psi_R \qquad (40)$$

Is this an acceptable result? Can the values of counterterms $V(p^2)$ and $\Sigma(q^2)$, be anything?
NO! Charge conservation $\longrightarrow e_R = e$.

Renormalization in PT: non-renormalization of e

non-ren of electric charge

e cannot be affected by the renormalization of QCD-induced divergencies!
In presence of strong interactions, makes the charge of the photon equal the sum of the charges of constituent quarks, in spite of the complex QCD dynamics that holds the quarks together

Renormalization is consistent with charge conservation if and only if

$$\frac{V(q^2)}{\Sigma(p^2)} = 1, \quad q^2 \longrightarrow 0. \tag{41}$$

Hold at all orders of PT. Fundamental constraint on the consistency of the theory.
Homework: Check 1-loop by evaluation of integrals in $V(q)$ and $\Sigma(p)$.

Renormalization program for QCD at 1-loop order

Calculate using dim reg

- quark and gluon self-energies
- coupling of quarks to gluon (QGV)
- self-coupling of gluons (TGV)
- ...

Start with a dim reg theory from the beginning!

Regularized and Renormalized QCD

Replace bare fields and couplings with renormalized ones (assume $m_q = 0$)

$$\psi_{bare} = Z_2^{1/2}\psi_R \qquad A^\mu_{bare} = Z_3^{1/2} A^\mu_R \qquad g_{bare} = Z_g \mu^\epsilon g_R$$

Note that we

- extracted dimensions out of g_{bare}
- introduced the dimensional parameter μ, *renormalization scale*
- prepared for the renormalized coupling g_R to be dimensionless when $D \to 4$

The QCD Lagrangian, becomes:

$$\begin{aligned} \mathcal{L} &= Z_2 \overline{\psi} i \partial\!\!\!/ \psi - \frac{1}{4} Z_3 F^a_{\mu\nu} F^{\mu\nu}_a + Z_g Z_2 Z_3^{1/2} \mu^\epsilon g \overline{\psi} A\!\!\!/ \psi \\ &+ (\text{gauge fixing, ghosts}, \cdots) \end{aligned} \tag{42}$$

Regularized and Renormalized QCD

It is customary to define

$$Z_1 = Z_g Z_2 Z_3^{1/2} \tag{43}$$

If we set $Z_n = 1 + \delta_n$, we then obtain:

$$\begin{aligned}\mathcal{L} &= \overline{\psi}\, i\, \not{\partial}\psi - \frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} + \mu^\epsilon g \overline{\psi}\not{A}\psi + [\text{ghosts, GM}] \\ &+ \delta_2\overline{\psi}\, i\, \not{\partial}\psi - \frac{1}{4}\delta_3 F^a_{\mu\nu}F_a^{\mu\nu} + \delta_1 \mu^\epsilon g \overline{\psi}\not{A}\psi \end{aligned} \tag{44}$$

The counter-terms δ_n are fixed by requiring the 1-loop Green functions to be finite.

Regularized and Renormalized QCD

Explicit evaluation of counterterms:

$$\text{quark self-energy:} \qquad \delta_2 = -C_F \left(\frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \right) \tag{45}$$

$$\text{gluon self-energy:} \qquad \delta_3 = \left(\frac{5}{3} C_A - \frac{4}{3} n_f T_F \right) \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \tag{46}$$

$$q\bar{q}g \text{ vertex corrections:} \qquad \delta_1 = -(C_A + C_F) \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \tag{47}$$

where $\alpha_s = g^2/4\pi$.

Regularized and Renormalized QCD

$$Z_g = \frac{Z_1}{Z_2 Z_3^{1/2}} = 1 + \delta_1 - \delta_2 - \frac{1}{2}\delta_3 \tag{48}$$

$$= 1 + \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\left[-\frac{11}{6}C_A + \frac{2}{3}n_f T_F\right] \equiv 1 - \frac{1}{\epsilon}\left(\frac{b_0}{2}\right)\alpha_s$$

- cancellation of C_F terms: quark self-energy (Z_2) and abelian vertex correction (Z_1)
- same as in the QCD non-renormalization of the electric coupling
- non-abelian vertex correction contributes TO the QCD coupling renormalization: consequence of gauge invariance
- individually, non-abelian contributions to self-energy and to vertex are not gauge-invariant, only their sum is.
- consistency of the renormalization procedure requires that ren g defining the intn strength of quarks and gluons, same as self-intn of gluons

α_s runs: renormalization scale independence

recall μ is tool to define coupling in D dim (dim reg) bare coupling g_{bare} insensitive to choice of μ

$$\frac{dg_{bare}}{d\mu} = 0 \tag{49}$$

Using the definition of g: $g_{bare} = \mu^{\epsilon} Z_g g$, we then get

$$\epsilon\mu^{2\epsilon}\ Z_g^2\ \alpha_s + \mu^{2\epsilon}\ 2\alpha_s\ Z_g \frac{dZ_g}{dt} + \mu^{2\epsilon}\ Z_g^2 \frac{d\alpha_s}{dt} = 0 \tag{50}$$

where

$$\frac{d}{dt} = \mu^2 \frac{d}{d\mu^2} = \frac{d}{d\log\mu^2} \tag{51}$$

Z_g depends upon μ only via the presence of α_s.

α_s runs: renormalization scale independence

Define

$$\beta(\alpha_s) \equiv \frac{d\alpha_s}{dt} \quad (52)$$

we then get:

$$\beta(\alpha_s) + 2\,\frac{\alpha_s}{Z_g}\,\frac{dZ_g}{d\alpha_s}\beta(\alpha_s) = -\epsilon\;\alpha_s \quad (53)$$

using eq. (49) and expanding in powers of α_s, we get:

$$\beta(\alpha_s) = \frac{-\epsilon\;\alpha_s}{1 + 2\,\frac{\alpha_s}{Z_g}\,\frac{dZ_g}{d\alpha_s}} = \frac{-\epsilon\;\alpha_s}{1 - \frac{b_0\alpha_s}{\epsilon}} = -b_0\alpha_s^2 + \mathcal{O}(\alpha_s^2, \epsilon) \quad (54)$$

and finally:

$$\beta(\alpha_s) = -b_0\alpha_s^2 \;\text{ with }\; b_0 = \frac{1}{2\pi}\left(\frac{11}{6}C_A - \frac{2}{3}n_f T_F\right) = \frac{1}{12\pi}(33 - 2n_f) \quad (55)$$

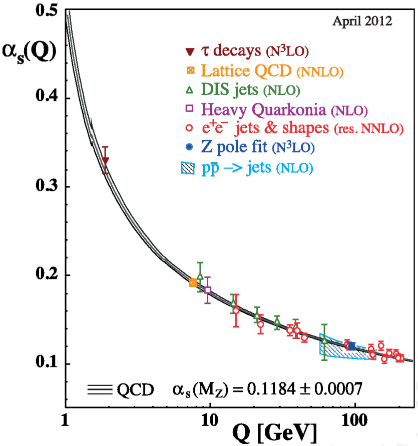
α_s runs: renormalization scale independence

We can now solve eq (52), assuming $b_0 > 0$ (which is true provided the number of quark flavours is less than 16) and get the famous *running* of α_s:

$$\alpha_s = \frac{1}{b_0 \log(\mu^2/\Lambda^2)} \tag{56}$$

The parameter Λ describes the boundary condition of the first order differential equation defining the running of α_s, and corresponds to the scale at which the coupling becomes infinity.

α_s runs: renormalization scale independence



running α_s everywhere

- $\alpha_s = \alpha_s(\mu^2)$ is OK. It is not an observable.
- α_s is at the heart of every QCD-related observable $\Gamma, \sigma, \ldots$
- observables as convolutions of $\alpha_s(\mu^2)$ and matrix elements: no μ-dependence, non-trivial cancellations!
- e.g. pQCD calculation of

$$\frac{\sigma(e^+e^- \longrightarrow hadrons)}{\sigma(e^+e^- \longrightarrow \mu^+\mu^-)} = \left(3\sum_f Q_f^2\right) R[\alpha_s(\mu^2), s/\mu^2]$$

R depends on μ explicity via $f_{(n)}$ and implicitly through α_s, as

$$R[\alpha_s, s/\mu^2] = 1 + \alpha_s f_1(t) + \alpha_s^2 f_2(t) + \cdots = \sum_{n=0}^{\infty} \alpha_s^n f_{(t)}(t) \quad (57)$$

where $t = s/\mu^2$.

running α_s and RGE (higher orders)

The formal proof of the previous result can be obtained by showing that the general form of equation

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s)\right] \; R\left(\alpha_s, \frac{s}{\mu^2}\right) = 0 \tag{58}$$

is given by

$$R\left(\alpha_s(s), 1\right) \quad \text{with} \quad \frac{d\alpha_s}{d\log\frac{s}{\mu^2}} = \beta(\alpha_s) \tag{59}$$

QCD in e^+e^- collisions

e^+e^- collisions

- provide one of the cleanest environments
- theoretical calculations have high accuracy
- experimental data with high precision (LEP, LEP2 and SLC)
- process $e^+e^- \to \gamma^*/Z^0 \to q\overline{q} \to$ hadrons
- provide an almost point-like source of quark pairs (vs hadron collisions)
- some high-multiplicity state of hadrons in the final state

If final state is NOT made of $q's$ and $g's$, but of $\pi's$, $K's$, $\rho's$, etc., how close to reality is the study of $q\overline{q}g\cdots g\cdots$ as final state?

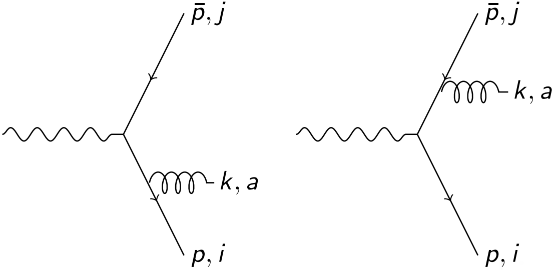
You'd be surprised!

pQCD in e^+e^- collisions: soft gluon emission

Build-up of final state hadron multiplicity needs seeds of soft gluons
Emitted with large probability: bremstahlung spectrum behaves like dE/E
soft $\equiv$ long wavelength $\longrightarrow$ insensitive to interactions on time scales shorter than their wavelength and to parton's spin
only sensitive to colour flow

pQCD in e^+e^- collisions: soft gluon emission

Consider soft-gluon emission in the $q\overline{q}$ decay of an off-shell photon:



$$
\begin{aligned}
A_{soft} &= \overline{u}(p)\not{\epsilon}(k)(ig\lambda^a_{ij})\frac{-i}{\not{p}+\not{k}}\Gamma^\mu v(\overline{p}) + \overline{u}(p)\Gamma^\mu\frac{i}{\not{\overline{p}}+\not{k}}(ig\lambda^a_{ij})\not{\epsilon}(k)v(\overline{p}) \\
&= g\ \lambda^a_{ij}\ \overline{u}(p)\left[\not{\epsilon}(k)\frac{(\not{p}+\not{k})}{2p\cdot k}\Gamma^\mu - \Gamma^\mu\frac{(\not{\overline{p}}+\not{k})}{2\overline{p}\cdot k}\not{\epsilon}(k)\right]\ v(\overline{p})
\end{aligned} \tag{60}
$$

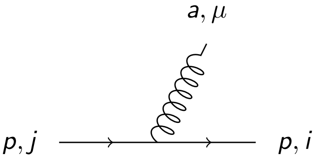
independent of the specific form factor Γ_μ

pQCD in e^+e^- collisions: soft gluon emission

Neglecting the factors of $\not{k}$ in the numerator (since soft gluons $k \ll p, \overline{p}$) and using Dirac eqs, we get:

$$A_{soft} \;=\; g\lambda^a_{ij}\left(\frac{p\cdot\epsilon}{p\cdot k} - \frac{\overline{p}\cdot\epsilon}{\overline{p}\cdot k}\right) A_{Born} \qquad (61)$$

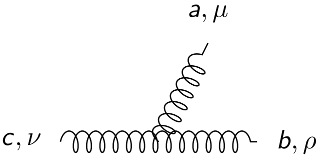
Soft gluon emission factorizes!
General Feynman rules for soft-gluon emission:



$$= \; g\;\lambda^a_{ij}\;2p^\mu \qquad (62)$$

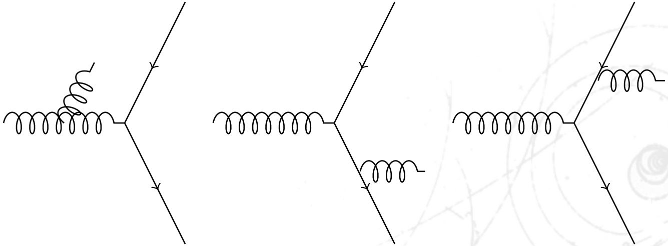
pQCD in e^+e^- collisions: soft gluon emission

In fact you can check that the $g \longrightarrow gg$ soft-emission rule is

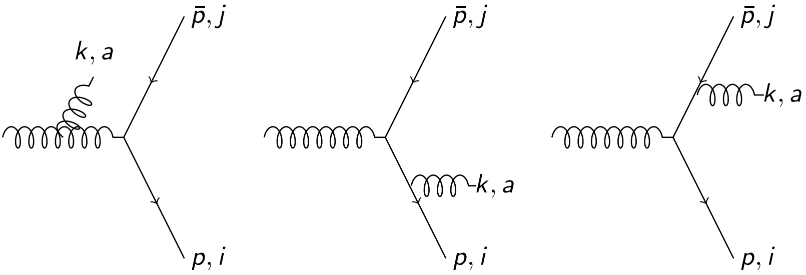


$$= igf^{abc}2p^{\mu}g^{\nu\rho} \qquad (63)$$

If we consider the "decay" of a virtual gluon into a quark pair. One more diagram should be added to those considered and quark pair is not in a colour-singlet state:



pQCD in e^+e^- collisions: soft gluon emission



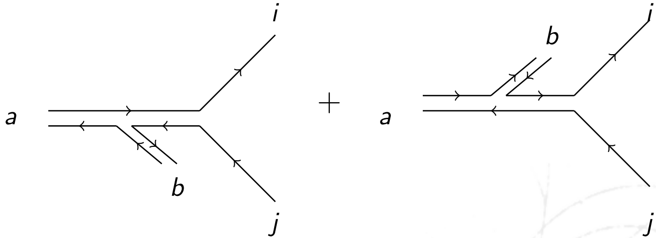
(64)

$$\stackrel{k\to 0}{=} \left\{ igf^{abc}\lambda^c_{ij}\left(\frac{Q\epsilon}{QK}\right) + g(\lambda^b\lambda^a)_{ij}\left(\frac{p\epsilon}{pk}\right) - g(\lambda^a\lambda^b)_{ij}\left(\frac{\overline{p}\epsilon}{pk}\right)\right\} A_{Born}$$

$$= \left\{ g(\lambda^a\lambda^b)_{ij}\left[\frac{Q\epsilon}{Qk} - \frac{\overline{p}\epsilon}{pk}\right] + g(\lambda^b\lambda^a)_{ij}\left[\frac{p\epsilon}{pk} - \frac{Q\epsilon}{Qk}\right]\right\} A_{Born} \qquad (65)$$

pQCD in e^+e^- collisions: soft gluon emission

The two factors correspond to the two possible ways colour can flow in this process:



(66)

Colour connection: $\bar{q} \to g^*$ and $q \to g^{soft}$ plus inverse
Altogether soft gluon emission from $\bar{q}$-line plus q-line

Squaring the total amplitude, and summing over initial and final-state colours, the interference between the two pieces is suppressed by $1/N^2$ relative to the individual squares:

$$\sum_{a,b,i,j} |(\lambda^a \lambda^b)_{ij}|^2 = \sum_{a,b} tr(\lambda^a \lambda^b \lambda^b \lambda^a) = \frac{N^2-1}{2} C_F = \mathcal{O}(N^3) \quad (67)$$

$$\sum_{a,b,i,j} |(\lambda^a \lambda^b)_{ij}||(\lambda^b \lambda^a)_{ij}|^* = \sum_{a,b} tr(\lambda^a \lambda^b \lambda^a \lambda^b) = \frac{N^2-1}{2} \underbrace{(C_F - \frac{C_A}{2})}_{-\frac{1}{2N}} = \mathcal{O}(N) \quad (68)$$

Emission of a soft gluon can be described at LO in N, as the incoherent sum of the emission from the two colour currents.

pQCD in e^+e^- collisions: angular ordering

Soft gluon emission drives the perturbative evolution of quarks !!!

Angular ordering:

continuous reductions of opening angle at which successive soft gluons are emitted by evolving quark
Radiation is confined within smaller and smaller cones around the quark direction $\rightsquigarrow$ collimated spray of partons, JETS!

- colour flow during the jet evolution forces $q\bar{q}$ pairs into a colour-singlet state and to stay close in phase-space
- pre-confinement of colour-singlet clusters!!

pQCD in e^+e^- collisions: angular ordering

How does colour ordering work? Use $\gamma^* \to q\bar{q}$ and factorized amp for soft gluon emission Eq.(61)

Squaring, summing over colours and including the gluon phase-space we get the following result:

$$\begin{aligned} d\sigma_g &= \sum |A_{soft}|^2 \frac{d^3k}{(2\pi)^3 2k^0} \\ &= g^2 C_F |A_{Born}|^2 \frac{-2p^\mu \bar{p}^\nu}{(pk)(\bar{p}k)} \sum \epsilon_\mu \epsilon^*_\nu \frac{d^3k}{(2\pi)^3 2k^0} \\ &= d\sigma_0 \; g^2 C_F \; \frac{2(p\bar{p})}{(pk)(\bar{p}k)} \left(\frac{d\phi}{2\pi}\right) \frac{k^0 dk^0}{8\pi^2} d\cos\theta \\ &= d\sigma_0 \; \frac{\alpha_s C_F}{\pi} \; \frac{dk^0}{k^0} \frac{d\phi}{2\pi} \frac{1-\cos\theta_{ij}}{(1-\cos\theta_{ik})(1-\cos\theta_{jk})} d\cos\theta \end{aligned} \tag{69}$$

where $\theta_{\alpha\beta} = \theta_\alpha - \theta_\beta$, and i, j, k refer to the q, $\bar{q}$ and gluon directions

pQCD in e^+e^- collisions: from a $q\bar{q}$ pair to hadrons

Partons forming a colour-singlet cluster will be close in phase-space because:

- colour flows directly from emitting parton to emitted one
- jets collimate
- radiation emitted at later stages is even softer

sets the scene for *hadronization* (non-perturbative process that will bind together colour-singlet parton pairs) to happen locally inside the jet so, NOT a collective process: only pairs of nearby partons are involved

pQCD in e^+e^- collisions: differential x-sec

Differential x-sec in $e^+e^- \longrightarrow q\bar{q}(g)$

$$d\sigma_R = \sigma_0 \frac{2\alpha_s}{\pi} \; C_F \; \frac{dk_0}{k_0} \; \frac{d\cos\theta}{1-\cos^2\theta} \text{ with } \sigma_0 = \text{Born amplitude} \tag{70}$$

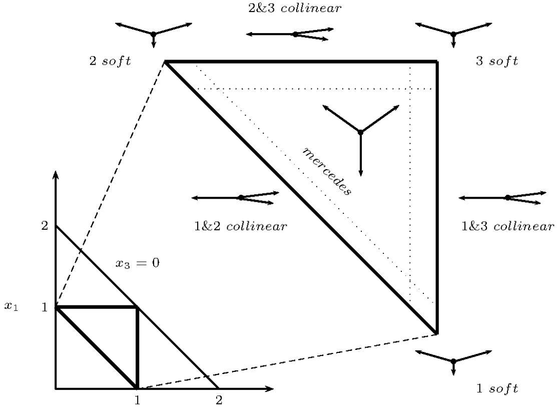
In the soft-g limit the q and $\bar{q}$ are back-to-back, and

$$q\cdot\bar{q} = 2q_0\bar{q}_0, \;\; q\cdot k = q_0 k_0 (1-\cos\theta), \;\; \bar{q}\cdot k = \bar{q}_0 k_0 (1+\cos\theta) \tag{71}$$

pQCD in e^+e^- collisions: differential x-sec

$d\sigma_g$ has soft and collinear singularities: cancel against virtual corr diag

KLN Theorem: for sufficiently inclusive observables the final states produced by the virtual and real emission diagrams in the soft or collinear limit are the same, and both contribute



x_i's space:
$0 \le x_i \le 1$ with
$x_3 = 2 - x_1 - x_2$

soft limit:
$x_i \to 0 \quad \forall\ i = 1, 2, 3$

collinear limit:
$x_1 \to 1 : \theta_{23} \to 0$
$x_2 \to 1 : \theta_{31} \to 0$
$x_3 \to 1 : \theta_{12} \to 0$

mercedes:
$\theta_{12} = \theta_{23} = \theta_{31} = \frac{\pi}{3}$
$x_1 = x_2 = x_3 = \frac{\sqrt{5}-1}{2}$

pQCD in e^+e^- collisions: jet rates

A jet definition/criteria:

particles distributed in sets of invariant mass smaller than a given parameter M, requiring that one particle only belongs to one jet, and that no other particles (or jets) can be added to a given jet without its mass exceeding M

For $q\bar{q}g$ 3-part final state:

- 3-jet events if $(q+k)^2$, $(\overline{q}+k)^2$ and $(q+\overline{q})^2$ are all larger than M^2
- 2-jet events when at least one of these quantities gets smaller than M^2
- e.g. emission of g near q, with $2qk = 2q^0k^0(1\cos\theta) < M^2$, one jet defined by $\overline{q}$, the other by the system $q+k$.

pQCD in e^+e^- collisions: 2- and 3- jet rate

Including Born contribution (only two jets!), we use parameter $y = M^2/s$, study jet multiplicity as function of y:

$$\begin{aligned} \sigma_{2-jet} &= \sigma_0 \left[1 - \frac{\alpha_s C_F}{\pi} \log^2 y + \cdots \right] \\ \sigma_{3-jet} &= \sigma_0 \frac{\alpha_s C_F}{\pi} \log^2 y + \cdots \end{aligned}$$

If $y \longrightarrow 0$, σ_{3jet} becomes larger than σ_{2jet} . If y is sufficiently small, we can even get $\sigma_{2jet} < 0$! HO corrections are important

pQCD in e^+e^- collisions: n-jet rate

What happens when more gluons are emitted? Assuming emission of a second gluon will also factorize (not true!), we can repeat the calculation at HO and get

$$
\begin{aligned}
\sigma_{2-jet} &= \sigma_0 \left[1 - \frac{\alpha_s C_F}{\pi} \log^2 y + \frac{1}{2!} \left(\frac{\alpha_s C_F}{\pi} \log^2 y \right)^2 + \cdots \right] \\
&= \sigma_0 e^{-\frac{\alpha_s C_F}{\pi} \log^2 y} \\
\sigma_{3-jet} &\sim \sigma_0 \frac{\alpha_s C_F}{\pi} \log^2 y \; e^{-\frac{\alpha_s C_F}{\pi} \log^2 y} \\
&\vdots \\
\sigma_{n+2-jet} &\sim \sigma_0 \frac{1}{n!} \left(\frac{\alpha_s C_F}{\pi} \log^2 y \right)^n \; e^{-\frac{\alpha_s C_F}{\pi} \log^2 y}
\end{aligned}
\tag{72}
$$

Poisson distribution! avg number of jets

$$
\langle n_{jet} \rangle \simeq 2 + \frac{\alpha_s C_F}{\pi} \log^2 y \tag{73}
$$

pQCD in e^+e^- collisions: n-jet rate

If $M \sim \Lambda_{QCD}$, each particle gets identified with an independent jet. Estimate the s-dependence of the avg particle multiplicity

$$\langle n_{jet} \rangle \sim \frac{\alpha_s C_F}{\pi} \log^2 \frac{s}{\Lambda^2} = \frac{C_F}{\pi} \frac{1}{b_0 \log \frac{s}{\Lambda^2}} \log \frac{s}{\Lambda^2} \simeq \frac{C_F}{\pi b_0} \log \frac{s}{\Lambda^2} \quad (74)$$

The final state particle multiplicity grows with $\log(s)$.
Real life: once 1st gluon is emitted, additional gluons can be emitted from it as well. Final-state multiplicity will be dominated by the emission of gluons from gluons. So in reality,

$$\langle n_{part}(s) \rangle \sim \exp \sqrt{\frac{2 C_A}{\pi b} \log \frac{s}{\Lambda^2}} \quad (75)$$

$$\langle n_{jet}(s) \rangle = 2 + 2 \frac{C_F}{C_A} \left(\cosh \sqrt{\frac{\alpha_s C_A}{2\pi} \log^2 \frac{1}{y}} - 1 \right) \sim \frac{C_F}{C_A} \exp \sqrt{\frac{\alpha_s C_A}{2\pi} \log^2 \frac{1}{y}} \quad (76)$$

pQCD in e^+e^- collisions: jet mass

Jet mass: divide the final state into two hemispheres, separated by the plane orthogonal to the thrust axis. We now call jets the two sets of particles on either side of the plane. The $\langle m^2 \rangle$ of the jet is then given by

$$\left\langle m_{jet}^2 \right\rangle = \frac{1}{2\sigma_0} \left[\int_{(\mathrm{I})} (q+k)^2 d\sigma_R + \int_{(\mathrm{II})} (\overline{q}+k)^2 d\sigma_R \right] \qquad (77)$$

The virtual correction does not enter here, since the pure $q\bar{q}$ final state has jet masses equal to 0. The result of this computation leads to

$$\left\langle m_{jet}^2 \right\rangle = \frac{\alpha_s C_F}{\pi} s \qquad (78)$$

pQCD in e^+e^- collisions: jet thrust

Another interesting variable often used in experimental studies is the thrust T, defined by:

$$T = \overset{max}{\hat{T}} \frac{\sum_i |\hat{p}_i \cdot \hat{T}|}{\sum_i |\vec{p}_i|}$$

where $\hat{T}$ is the thrust axis, defined so as to maximize T

In 3-body final states, $\hat{T}$ direction of the highest-energy parton, so T is proportional to twice its energy:

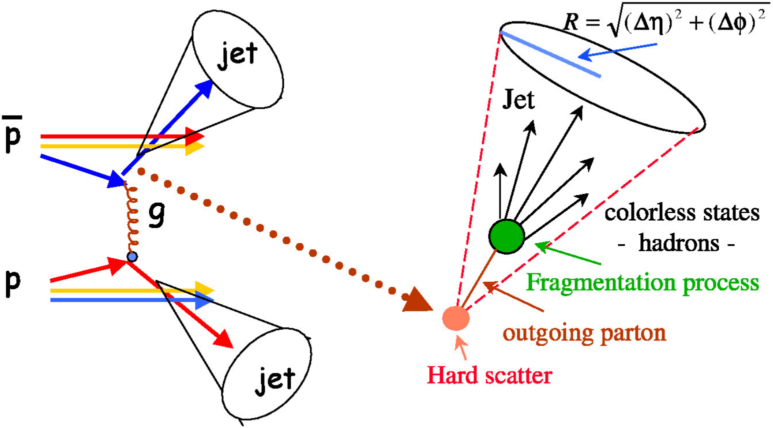
$$T = 2\frac{\overline{q}_0}{\sqrt{s}} \equiv 1 - \frac{(q+k)^2}{s} = 1 - \frac{m_{jet}^2}{s} \tag{79}$$

As a result:

$$\langle 1 - T \rangle = \frac{\alpha_s C_F}{\pi} \tag{80}$$

At LEP, $\langle 1 - T \rangle \simeq \frac{0.120}{\pi} \times \frac{4}{3} \simeq 0.05$.

Jets are experimental signatures of q's and g's



PDG: x-sec review and qcd review

QCD and the proton structure at large Q^2

Key ingredient to predict what will happen when hadrons collide: Know your hadron's structure!

q's and g's inside hadrons induce ALL had processes, event those of EW nature (such as W/Z boson production)!

Brief and qualitative discussion of:

- Can *pQCD* + *partons* be used to describe hadrons?
- How are partons *distributed* inside the proton?
- How does this *parton distribution* change with all relevant variables?