Notes on the Higgs potential

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Abstract

Please join me into the greatest adventure of all: exploring the unknown. This time we'll dive into the deep ocean of knowledge and explore the different aspects of the classical Higgs potential.

1 The deep ocean

1.1 Warming up: Global and local $U(1)$ invariance

Let us begin with the Lagrangian of a complex scalar field,[1]

$$\mathcal{L}_{scalar} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - V(\Phi^\dagger, \Phi) \tag{1}$$

where[2]

$$V(\Phi^\dagger, \Phi) = \mu^2 \Phi^\dagger \Phi + \lambda \left(\Phi^\dagger \Phi \right)^2 . \tag{2}$$

The first thing we can notice here is that this Lagrangian possesses an accidental $U(1)$ global symmetry,

$$\Phi \to e^{i\alpha} \Phi. \tag{3}$$

To make it local the gauge invariance principle tells us that we need to change the ordinary derivatives for the covariant ones,

$$\partial_\mu \to D_\mu = \partial_\mu + i g' Y B_\mu, \tag{4}$$

[1] A complex scalar and singlet field is written as $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$.

[2] At this point we still do not know what is the correct choice of signs for the μ and λ parameters.

where the vector field is invariant under gauge transformations,

$$B_\mu \to B_\mu - \partial_\mu \Gamma(x), \tag{5}$$

and the new local phase transformation is given by $e^{ig'Y\Gamma(x)}$.[3]

On the other hand, to imply that our new vector field B_μ behaves as a physical field, that is, that it also propagates through spacetime then we need to introduce its corresponding kinetic term. Therefore, our Lagrangian invariant under local $U(1)_Y$ transformations ought to be given by,

$$\mathcal{L}_{U(1)} = D_\mu \Phi^\dagger D^\mu \Phi - V(\Phi^\dagger, \Phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{6}$$

where $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$.[4] We have not added a mass term for the vector field as this will break our gauge invariance. This Lagrangian is the one employed within the topic of Scalar Electrodynamics.

1.2 Global and local $SU(2)$ invariance

Let us now consider a larger rank symmetry group. The smallest irreducible representations in the case of $U(1)$ were one-dimensional, as this group is Abelian, and that represented no problem. However, if we now consider the case of $SU(2)$, things get a little bit more complicated, but not too much.

Without any known reason *a priori* all the fields in the Standard Model are always assigned to either the fundamental or the trivial representation of the local symmetry groups. In this sense, even though we could assign our scalar field to, a triplet of $SU(2)$ for example, Nature has chosen to only consider it as a doublet. Therefore, from now on our scalar field will be given as a two-dimensional complex vector,

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \tag{7}$$

where ϕ_i are real scalar fields.

In general, we can apply a special unitary and global transformation,

$$U = \begin{pmatrix} e^{i\alpha} \cos\theta & e^{-i\beta} \sin\theta \\ -e^{i\beta} \sin\theta & e^{-i\alpha} \cos\theta \end{pmatrix} \tag{8}$$

[3]Show that the Lagrangian is indeed invariant under the aforementioned transformation.

[4]Show that the kinetic term is also invariant under the gauge transformation $B'_\mu = B_\mu - \partial_\mu \Gamma(x)$.

to Φ and nevertheless leave invariant the product,

$$\Phi'^{\dagger}\Phi' = \Phi^{\dagger}U^{\dagger}U\Phi = \Phi^{\dagger}\Phi. \tag{9}$$

Regarding this we consider that its Lagrangian (see Eq. (1)) is invariant under $SU(2)$ transformations. Furthermore, we can also see that this Lagrangian still preserves untouched the invariance under the $U(1)$ global transformations. In fact, this is telling us that the accidental global symmetry group that a Lagrangian for a complex doublet has is, in general, an $U(2)$.[5]

Let us go back to $SU(2)$. It has three generators which are the well known Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{10}$$

Application of the gauge invariance principle means[6]

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu + i\frac{g}{2}\sigma_i W_\mu^i, \tag{11}$$

that is, the introduction of a gauge boson per each generator. The Lagrangian now takes the form,

$$\mathcal{L}_{SU(2)} = D_\mu\Phi^{\dagger}D^\mu\Phi - V(\Phi^{\dagger},\Phi) - \frac{1}{4}F_{\mu\nu}^i F_i^{\mu\nu}, \tag{12}$$

where $F_{\mu\nu}^i = \partial_\mu W_\nu^i - \partial_\nu W_\mu^i + g\epsilon^{ijk}W_\mu^j W_\nu^k$ (the last term is a consequence of the Non-Abelian nature of $SU(2)$).

1.3 The Electroweak Theory: $SU(2)_L \times U(1)_Y$

We now consider a Lagrangian invariant under local $SU(2)\times U(1)$ transformations,

$$\mathcal{L}_{SU(2)\times U(1)} = D_\mu\Phi^{\dagger}D^\mu\Phi - V(\Phi^{\dagger},\Phi) - \frac{1}{4}F_{\mu\nu}^i F_i^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}, \tag{13}$$

where $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ and $F_{\mu\nu}^i = \partial_\mu W_\nu^i - \partial_\nu W_\mu^i + g\epsilon^{ijk}W_\mu^j W_\nu^k$. Furthermore, the covariant derivative is written as,

$$D_\mu = \partial_\mu + i\frac{g}{2}\sigma_i W_\mu^i + ig'YB_\mu. \tag{14}$$

[5]The main difference between $U(2)$ and $SU(2)$ is that the determinant of the first one is a phase whereas for the second one is +1.

[6]It is common to denote $\tau_i \equiv \frac{\sigma_i}{2}$.

Explicitly,

$$D_\mu = \begin{pmatrix} \partial_\mu + i\frac{g}{2}W_\mu^3 + ig'YB_\mu & i\frac{g}{2}(W_\mu^1 - iW_\mu^2) \\ i\frac{g}{2}(W_\mu^1 + iW_\mu^2) & \partial_\mu - i\frac{g}{2}W_\mu^3 + ig'YB_\mu \end{pmatrix}. \tag{15}$$

Charge is defined through,

$$Q = \frac{T_3}{2} + Y = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix}. \tag{16}$$

We choose the hypercharge, Y, in order to have

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \tag{17}$$

that is $Y = 1/2$.

1.4 Spontaneous Symmetry Breaking

Let us finally study the Higgs potential. For that we will investigate it through its real components,

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}. \tag{18}$$

The scalar potential takes the form,[7]

$$V(\phi_1, \phi_2, \phi_3, \phi_4) = \frac{\mu^2}{2}\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right) + \frac{\lambda}{4}\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right)^2. \tag{19}$$

A stationary point requires,

$$\frac{\partial V}{\partial \phi_i}\bigg|_{<\phi_i>} = 0. \tag{20}$$

In order to preserve conservation of electric charge we need to consider that $< \phi_3 >\neq 0$ while the rest are zero. We then have, that three of the four are already identically zero

$$\frac{\partial V}{\partial \phi_i}\bigg|_{v_i=<\phi_i>=0} = 0. \qquad i = 1, 2, 4. \tag{21}$$

[7] Notice how the scalar potential is invariant under an $SO(4)$ transformation. This symmetry group is called the custodial symmetry group, which explains why the ρ parameter, to appear next, is one.

On the other hand,

$$\frac{\partial V}{\partial \phi_3}\Big|_{v_3=<\phi_3>\neq 0} = \mu^2 v_3 + \lambda v_3^3 = 0, \tag{22}$$

which implies, apart from the trivial case $v_3 = 0$,

$$v_3 = \sqrt{-\frac{\mu^2}{\lambda}}. \tag{23}$$

In order to have v_3 real (we want a CP conserving scalar potential), we require either $\lambda < 0$ or $\mu^2 < 0$. Let us see the consequence of each of these possible cases,

$$J = \frac{\partial^2 V}{\partial \phi_3^2}\Big|_{v_3} = \mu^2 + 3\lambda v_3^2 = -2\mu^2. \tag{24}$$

We see that if λ was the one negative then $J < 0$ and our critical point would have to be a local maximum. On the other hand, if $\mu^2 < 0$ then $J > 0$ and we have a local minimum.

In fact, from the Higgs mass matrix which emerges from expanding V about its minimum,

$$V = V(min) + \frac{1}{2}(\phi - \phi_0)_i(\phi - \phi_0)_j \frac{\partial^2 V}{\partial \phi_i \phi_j}\Big|_{min}. \tag{25}$$

We denote it and then compute it as

$$\mathbf{m}_{ij}^2\Big|_{min} = \frac{\partial^2 V}{\partial \phi_i \phi_j}\Big|_{min} \tag{26}$$

$$\rightarrow \mathbf{m}^2\Big|_{min} = \begin{pmatrix} v_3^2\lambda + \mu^2 & 0 & 0 & 0 \\ 0 & v_3^2\lambda + \mu^2 & 0 & 0 \\ 0 & 0 & 3v_3^2\lambda + \mu^2 & 0 \\ 0 & 0 & 0 & v_3^2\lambda + \mu^2 \end{pmatrix}\Bigg|_{min} \tag{27}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2\mu^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{28}$$

one concludes that the only possibility is to have $\lambda > 0$ with $\mu^2 < 0$. From here it can also be seen how three degrees of freedom are massless while only one is massive. This massive state is the Higgs state. The rest are called Goldstone bosons.

Thus, the correct form of our Higgs potential shall be,

$$V = -\mu^2\Phi^\dagger\Phi + \lambda\left(\Phi^\dagger\Phi\right)^2. \tag{29}$$

We consider perturbations around the minimum which, by the way, we have shifted,

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_1 + i\phi_2 \\ (v_3 + h(x)) + i\phi_4 \end{pmatrix}. \tag{30}$$

Now, each field of the four has a null vev.

Also, it can be noticed that our global symmetry has now been broken to $SO(3)$. That is $SO(4) \simeq SU(2)_a \times SU(2)_b \to SO(3) \simeq SU(2)_{a=b}$.

Note how $\Phi^\dagger\Phi$ has an $SU(2)$ symmetry while in term of its real components it is $SO(4) \sim SU(2)\times SU(2)$. Spontaneous Symmetry Breaking means that one has for the real scalar fields a remnant $SO(3) \sim SU(2)$. That is it still preserves the symmetry group which originally had $\Phi^\dagger\Phi$. This is the role of the custodial symmetry.

1.5 Massive vector fields and a masless one

Let us now study the covariant derivative of the scalar field,

$$D_\mu\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} \partial_\mu + i\frac{g}{2}W_\mu^3 + ig'YB_\mu & i\frac{g}{2}(W_\mu^1 - iW_\mu^2) \\ i\frac{g}{2}(W_\mu^1 + iW_\mu^2) & \partial_\mu - i\frac{g}{2}W_\mu^3 + ig'YB_\mu \end{pmatrix}\begin{pmatrix} \phi_1 + i\phi_2 \\ (v_3 + h(x)) + i\phi_4 \end{pmatrix} \tag{31}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \partial_\mu(\phi_1 + i\phi_2) + i(\frac{g}{2}W_\mu^3 + g'YB_\mu)(\phi_1 + i\phi_2) + i\frac{g}{2}(W_\mu^1 - iW_\mu^2)[(v_3 + h(x)) + i\phi_4] \\ \partial_\mu[(v_3 + h(x)) + i\phi_4] - i(\frac{g}{2}W_\mu^3 - g'YB_\mu)[(v_3 + h(x)) + i\phi_4] + i\frac{g}{2}(W_\mu^1 + iW_\mu^2)(\phi_1 + i\phi_2) \end{pmatrix}. \tag{32}$$

After the product with its Hermitian conjugate and considering $Y = 1/2$,

$$|D_\mu\Phi|^2 = \begin{pmatrix} \frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}(\partial\phi_2)^2 + \frac{v_3^2 g^2}{4}W_\mu^+ W^{\mu,-} + \cdots \\ \frac{1}{2}(\partial h)^2 + \frac{1}{2}(\partial\phi_4)^2 + \frac{v_3^2}{8}(\frac{g}{2}W_\mu^3 - g'YB_\mu)^2 + \cdots \end{pmatrix}, \tag{33}$$

where we have denoted by $W^\pm = \frac{W^1 \mp iW^2}{\sqrt{2}}$.[8] In order to define new states the following can be done,

$$\begin{pmatrix} W^3 & B \end{pmatrix}_\mu \frac{v_3^2}{8}\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}\begin{pmatrix} W^3 \\ B \end{pmatrix}^\mu \Rightarrow \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w W_\mu^3 + \sin\theta_w B_\mu \\ -\sin\theta_w W_\mu^3 + \cos\theta_w B_\mu \end{pmatrix}. \tag{34}$$

[8] The choice in signs must be done according to the values of hypercharge to fermions.

The mass matrix is rank one with eigenvalues $M_A^2 = 0$ (massless photon) and $M_Z^2 = \frac{v_3^2}{4}(g^2 + g'^2)$. One can show that the angle of rotation satisfies

$$\cos\theta_w = \frac{g}{\sqrt{g^2 + g'^2}}, \qquad \sin\theta_w = \frac{g'}{\sqrt{g^2 + g'^2}}. \tag{35}$$

If we also define that $M_W^2 = v_3^2 g^2/4$ then we see that[9]

$$\rho \equiv \frac{M_W}{M_Z \cos\theta_w} = 1. \tag{36}$$

Thus, the kinetic term of our doublet scalar field is

$$|D_\mu \Phi|^2 = \begin{pmatrix} \frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}(\partial\phi_2)^2 + M_W^2 W_\mu^+ W^{\mu,-} + \cdots \\ \frac{1}{2}(\partial h)^2 + \frac{1}{2}(\partial\phi_4)^2 + \frac{M_Z^2}{2} Z_\mu Z^\mu + \cdots \end{pmatrix}. \tag{37}$$

Only the scalar and real field h is massive whereas the rest stays massless. The massless fields are called the Goldstone bosons which have actually become the longitudinal components of the new massive vector fields.

1.6 A real triplet instead of a doublet: ρ parameter

Let us consider, instead of a Higgs doublet, a real Higgs triplet,

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{pmatrix}. \tag{38}$$

Here the three generators (three dimensional representation) are,

$$\tau_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \tau_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{39}$$

In order to have the second component of Φ without electric charge we need to choose the hypercharge as $Y = 0$.

The Higgs potential acquires the form,

$$V(\Phi^\dagger, \Phi) = -\frac{\mu^2}{2}\left(\Sigma^{+2} + \Sigma^{02} + \Sigma^{-2}\right) + \frac{\lambda}{4}\left(\Sigma^{+2} + \Sigma^{02} + \Sigma^{-2}\right)^2. \tag{40}$$

[9] It is instructive to investigate this same parameter under a triplet instead of a scalar doublet.

Notice how it has an accidental and global $SO(3)$ symmetry.

The minimization can be achieved through a similar treatment as before,

$$< \Sigma^0 >= \sqrt{\frac{\mu^2}{\lambda}}. \tag{41}$$

We then do the following change,

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \Sigma^+ \\ v_0 + h(x) \\ \Sigma^- \end{pmatrix}, \tag{42}$$

where $< \Sigma^0 >= v_0/\sqrt{2}$. At this point, the initial global symmetry group, $SO(3) \simeq SU(2)$, has been broken to $SO(2)$.

Let us build the covariant derivative of Φ,

$$D_\mu \Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \partial_\mu + igW_\mu^3 & i\frac{g}{\sqrt{2}}(W_\mu^1 - iW_\mu^2) & 0 \\ i\frac{g}{\sqrt{2}}(W_\mu^1 + iW_\mu^2) & \partial_\mu & i\frac{g}{\sqrt{2}}(W_\mu^1 - iW_\mu^2) \\ 0 & i\frac{g}{\sqrt{2}}(W_\mu^1 + iW_\mu^2) & \partial_\mu - igW_\mu^3 \end{pmatrix} \begin{pmatrix} \Sigma^+ \\ v_0 + h(x) \\ \Sigma^- \end{pmatrix}. \tag{43}$$

One can easily show that computation of $|D\Phi|^2$ means now $M_W = gv_0/2$ and $M_Z = 0$. Here we may identify $Z = W^3$. Therefore, the ρ parameter is here given as

$$\rho \sim \frac{M_W}{M_Z} \to \infty. \tag{44}$$

1.7 Beyond the Standard Model: Inert Two Higgs Doublet Model

Let us consider the addition of a second Higgs doublet, Φ_2. This Higgs will not couple to the fermions but only to the Higgs of the SM (which we will denote from now on as Φ_1). In order to forbid such couplings we need to introduce a symmetry. We consider the simplest option to be Z_2 as is the smallest symmetry group. We do the following charge assignment under Z_2,

$$\Phi_1 \to +\Phi_1, \qquad \Phi_2 \to -\Phi_2, \tag{45}$$

$$\psi_L \to +\psi_L, \qquad \psi_R \to +\psi_R. \tag{46}$$

This means that in the Higgs potential terms with an odd number of Φ_2 will be forbidden.

The scalar potential is given by,

$$V_{\text{IDM}} = -\mu_1^2\Phi_1^\dagger\Phi_1 - \mu_2^2\Phi_2^\dagger\Phi_2 + \lambda_1(\Phi_1^\dagger\Phi_1)^2 + \lambda_2(\Phi_2^\dagger\Phi_2)^2 + \lambda_3(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) \tag{47}$$

$$+\lambda_4(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_5\left[(\Phi_1^\dagger\Phi_2)^2 + (\Phi_2^\dagger\Phi_1)^2\right]. \tag{48}$$

Recall that each doublet can be written in terms of its real fields,[10]

$$\Phi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2\\ \phi_3 + i\phi_4\end{pmatrix}, \tag{49}$$

and

$$\Phi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_5 + i\phi_6\\ \phi_7 + i\phi_8\end{pmatrix}. \tag{50}$$

Let us now consider the minimization conditions. We want our minimum to satisfy,

$$\frac{\partial V}{\partial \phi_i} = 0 \tag{51}$$

and to have the form,[11]

$$<\phi_i>= 0 \qquad (i \neq 3,7) \tag{52}$$

$$<\phi_j>= v_j \qquad (j = 3,7). \tag{53}$$

This implies,

$$v_3(2v_3^2\lambda_1 + v_7^2(\lambda_3 + \lambda_4 + 2\lambda_5) - 2\mu_1^2) = 0 \tag{54}$$

$$v_7(2v_7^2\lambda_2 + v_3^2(\lambda_3 + \lambda_4 + 2\lambda_5) - 2\mu_2^2) = 0. \tag{55}$$

For $v_3, v_7 \neq 0$ one can solve the latter system of equations and find v_3 and v_7 in terms of parameters of the potential,

$$\left\{\text{v3} \to \frac{\sqrt{\frac{(\lambda3+\lambda4+2\lambda5)\left(-2\lambda2\mu1^2+(\lambda3+\lambda4+2\lambda5)\mu2^2\right)}{-4\lambda1\lambda2+(\lambda3+\lambda4+2\lambda5)^2}}}{\sqrt{\frac{\lambda3+\lambda4}{2}+\lambda5}}, \text{v7} \to \frac{\sqrt{-2(\lambda3+\lambda4+2\lambda5)\mu1^2+4\lambda1\mu2^2}}{\sqrt{4\lambda1\lambda2-(\lambda3+\lambda4+2\lambda5)^2}}\right\}. \tag{56}$$

One could also study the masses for this set of 8 real fields by computing $\mathbf{m}^2$. One obtains that 5 of them are massive while 3 are massless, which is what one expects. Good.

[10] Realize that 8 real fields imply in our notation 4 of them to be charged and the other 4 to be neutral.

[11] Why? Investigate it.