All static circularly symmetric perfect fluid solutions of 2+1 gravity

Alberto A. García* and Cuauhtemoc Campuzano†
Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, Apartado Postal 14-740, 07000 México DF, Mexico
(Received 5 November 2002; published 27 March 2003)

Via a straightforward integration of the Einstein equations with a cosmological constant, all static circularly symmetric perfect fluid $2+1$ solutions are derived. The structural functions of the metric depend on the energy density, which remains, in general, arbitrary. Spacetimes for fluids satisfying linear and polytropic state equations are explicitly derived; they describe, among others, stiff matter, monatomic and diatomic ideal gases, nonrelativistic degenerate fermions, incoherent and pure radiation. As a by-product, we demonstrate the uniqueness of the constant energy density perfect fluid within the studied class of metrics. A full similarity of the perfect fluid solutions with a constant energy density of the $2+1$ and $3+1$ gravities is established.

DOI: 10.1103/PhysRevD.67.064014 PACS number(s): 04.20.Jb, 04.40.Nr

I. INTRODUCTION

In the last two decades a number of researches have been developed in $2+1$ gravity: the search for exact solutions, the quantization of fields coupled to gravity, topological aspects, black hole physics, and so on. The literature in this respect is extremely vast, for references on related works, see Ref. [1].

In the framework of exact solutions, from the beginning until now, attention has been focused on the search and the study of physically relevant solutions and models with sources: for instance, static *N*-body spaces [2–4], static and stationary metrics coupled to electromagnetic fields [5–10], scalar-dilaton fields [11], cosmology [12], perfect fluids [13–17], among others.

We believe that the physical relevance of $2+1$ solutions consists in the fact that they may provide a link or correspondence with solutions or physical aspects of the relevant $3+1$ gravity theory. For instance, the $2+1$ Bañados-Teitelboim-Zanelli (BTZ) black hole solution has been demonstrated to be a dimensional reduction of $3+1$ Plebański-Carter[A] spacetime [18]. Moreover, recently we demonstrated that conformally flat stationary axisymmetric spacetimes allow for a branch of metrics which can be thought of as the product $\mathbb{R}\times$BTZ [19]. It is worth mentioning that the structure $\mathbb{R}\times$BTZ with specific structural functions also has been found in the description of black holes on branes [20]. Concerning $2+1$ gravity coupled to perfect fluids, the results reported in [14] are remarkable: homogeneous isotropic Robertson-Walker cosmologies and the spherical star models, where comparisons to results predicted by the corresponding Newtonian theory were carried out. Incidentally, the Einstein equations for $3+1$ and $2+1$ Robertson-Walker cosmologies for a linear state equation can be solved, in general [21], therefore one can establish a full correspondence between solutions determined for these different dimensions; of course, their physical content depends on the dimension, such as, for example, a $2+1$ radiation-dominated universe is equivalent to $3+1$ dust-filled space, in agreement with [14].

In $3+1$ gravity, spherically symmetric perfect fluid metrics have been studied as candidates to model stars and celestial configurations. Long since, in spite of the great amount of work done in this direction, even in the static case the achievements made so far occur to be highly modest, mainly for lack of a general procedure to solve the nonlinear Einstein equations coupled to fluids governing the problem. Few exact physically interesting solutions can be listed: among them, the Schwarzschild perfect fluid solution and the Tolman dust model [22]. Most of the solutions to the mentioned equations lack deep physical interpretations; for original sources see [23]. There is no general $3+1$ solution for general linear or polytropic state equations; in these cases one has to use numerical techniques to derive numerical results.

On the contrary, in the static $2+1$ case one is able to determine, and this is the purpose of this paper, all static circularly symmetric spacetimes with the cosmological constant coupled to perfect fluids with and without zero pressure surfaces. In this context, some progress has been previously achieved. Cornish and Frankel [15] derived all universes obeying a polytropic state equation. Nevertheless, since their solutions were derived for zero cosmological constant, there is no way to determine a zero pressure surface. Hence these solutions extend to the whole spacetime, and consequently they are cosmological solutions. On the other hand, Cruz and Zanelli [16] established some consequences arising from the hydrostatic equilibrium Oppenheimer-Volkov equation, and derived for this equation a single solution for constant energy density; it should be pointed out that the expression of their g_{tt}-metric component has a misprint which is corrected in this work. By the way, we demonstrate here that the perfect fluid solution with constant energy density is the only conformally flat—in the sense of the vanishing of the Cotton tensor [24]—circularly symmetric solution. These analytical results, in our opinion, can shed some light on the search of other physically meaningful perfect fluid solutions of the $3+1$ gravity even when numerical methods are used.

In Sec. II, by a straightforward integration of the Einstein equations, we derive the general solution for the static circularly symmetric $2+1$ metric with a cosmological constant coupled to a perfect fluid solution with variable density ρ and pressure p.

*Electronic address: aagarcia@fis.cinvestav.mx
†Electronic address: ccvargas@fis.cinvestav.mx

Section III is devoted to represent this whole class of spacetimes in a canonical coordinate system. For a given equation of state of the form $p=p(\rho)$, certain particular families of perfect fluid solutions are derived; as concrete examples, the subcases of fluids obeying the linear law $p=\gamma\rho$, and those fluids subjected to a polytropic law $p=C\rho^{\gamma}$ in detail are derived.

In Sec. IV, from the Oppenheimer-Volkov equation certain properties of the studied solutions are established, for instance, for positive pressure p and positive density ρ, obeying a state equation $p=p(\rho)$, a microscopically stable fluid possesses a monotonically decreasing energy density, and vice versa.

In Sec. V, to facilitate the comparison of the interior Schwarzschild $3+1$ solution with the cosmological constant, the perfect fluid $2+1$ solution with $\rho=$const is derived. With this aim in mind, we search for an adequate representation of the corresponding structural functions and related quantities of these $3+1$ and $2+1$ spacetimes. A comparison table is presented. Via a dimensional reduction of the interior Schwarzschild with λ solution, the perfect fluid $2+1$ solution with constant ρ is obtained.

Finally, we end with some concluding remarks.

Einstein equations for 2+1 static circularly symmetric perfect fluid metric

As far as we know, in most of the publications dealing with the search of perfect fluid solutions in $2+1$ gravity—see, for instance, [13,15,16]—the energy-momentum conservation, i.e., the Oppenheimer-Volkov equation, has been used as a clue to obtain the desired results. On the contrary, we prefer to solve directly the corresponding Einstein equations; in such a case the energy-momentum conservation equations hold trivially.

The line element of static circularly symmetric $2+1$ spacetimes, in coordinates $\{t,r,\theta\}$, is given by

$$ds^2=-N(r)^2dt^2+\frac{dr^2}{G(r)^2}+r^2d\theta^2. \tag{1}$$

Note that we are using units such that the velocity of light $c=1$.

The Einstein equations with the cosmological constant for a perfect fluid energy-momentum tensor T_{ab},

$$\begin{aligned} G_{ab}&=R_{ab}-\tfrac{1}{2}g_{ab}R=\kappa T_{ab}-\lambda g_{ab},\\ T_{ab}&=(p+\rho)u_a u_b+p\,g_{ab},\quad u_a=-N\delta_a^t,\\ R&=2\kappa\rho-4\kappa p+6\lambda, \end{aligned}\tag{2}$$

for the metric (1) explicitly amount to

$$G_{tt}=-\frac{N^2}{2r}\frac{dG^2}{dr}=N^2(\kappa\rho+\lambda),$$

$$G_{rr}=\frac{1}{rN}\frac{dN}{dr}=\frac{1}{G^2}(\kappa p-\lambda),$$

$$G_{\theta\theta}=\frac{r^2}{N}\left[G^2\frac{d^2N}{dr^2}+\frac{1}{2}\frac{dN}{dr}\frac{dG^2}{dr}\right]=r^2(\kappa p-\lambda). \tag{3}$$

Notice that the combination of the Einstein equations $r^2G^2G_{rr}-G_{\theta\theta}=0$, for $N(r)\neq$const, gives rise to an important equation:

$$G\frac{d^2N}{dr^2}+\frac{dN}{dr}\left(\frac{dG}{dr}-\frac{G}{r}\right)=0,$$

$$\rightarrow\frac{d}{dr}\left(\frac{G}{r}\frac{dN}{dr}\right)=0, \tag{4}$$

which will be extensively used throughout this paper.

II. 2+1 PERFECT FLUID SOLUTION WITH VARIABLE $\rho(r)$

In this section, we derive the most general static circularly symmetric solution via a straightforward integration of the Einstein equations with λ for a perfect fluid. It is easy to establish that the structural functions $G(r)$ and $N(r)$ can be integrated in quadratures.

Integrating G_{tt}, Eq. (3), one arrives at

$$G(r)^2=-\lambda r^2-2\kappa\int_0^r r\rho(r)dr\equiv C-\lambda r^2-2\kappa\int^r r\rho(r)dr, \tag{5}$$

where C is an integration constant in which we have incorporated the constant value of the integral at the lower integration limit $r=0$, thus the remaining integral depends on the upper integration limit r; we use the r notation for the upper integration limit as well as to denote the integration variable. This convention will be used hereafter. From the second relation of Eq. (4), one obtains

$$\frac{dN}{dr}=n_1\frac{r}{G(r)}, \tag{6}$$

therefore

$$N(r)=n_1\int_0^r\frac{r}{G(r)}dr\equiv n_0+n_1\int^r\frac{r}{G(r)}dr. \tag{7}$$

The evaluation of the pressure $p(r)$ yields

$$\kappa p(r)=\frac{1}{N(r)}[n_1G(r)+\lambda N(r)]. \tag{8}$$

The metric (1), with $G(r)$ from Eq. (5), and $N(r)$ from Eq. (7), determines the general static circularly symmetric $2+1$ solution of the Einstein equations (3) with λ, positive or negative, for a perfect fluid, characterized by a pressure given by Eq. (8), and an arbitrary density $\rho(r)$. The fluid velocity is aligned along the timelike Killing direction ∂_t. In the derivation of the obtained solutions no positivity conditions were imposed, thus these fluids allow for negative p

and ρ. Nevertheless, to deal with realistic matter distributions one has to impose positivity conditions on the density, $\rho>0$, and the pressure, $p>0$, requiring additionally that $\rho>p$.

For a finite distributed fluid, the pressure p becomes zero at the boundary, say, $r=a$; this value of the radial coordinate r is determined as a solution of the equation $p(r)=0$.

For a nonvanishing cosmological constant, assuming that the values of the structural functions at the boundary $r=a$ are $N(a)$ and $G(a)$, the vanishing at $r=a$ of the pressure $p(r)$, given by Eq. (8), requires $n_1=-\lambda N(a)/G(a)$, hence

$$\kappa p(r)=\frac{\lambda}{N(r)G(a)}[N(r)G(a)-N(a)G(r)]. \tag{9}$$

If one is interested in matching the obtained perfect fluid metric with a vacuum metric with cosmological constant λ, the plausible choice at hand is the anti–de Sitter metric, with $\lambda=-1/l^2$, see Sec. IV, for which $G(a)=N(a)=\sqrt{-M_\infty+a^2/l^2}$ at the boundary $r=a$. Incidentally, for a nonzero cosmological constant, there is no room for dust. The zero character of the pressure yields the vanishing of the density, and consequently the metric describes the (anti–)de Sitter spacetime.

The structural functions $G(r)$ [Eq. (5)], $N(r)$ [Eq. (7)], and the pressure $p(r)$ [Eq. (8)] for any given density $\rho(r)$ determine the general solution of the $2+1$ Einstein equations with λ coupled to a perfect fluid for the general metric (1). As particular branches of solutions, we shall derive the solution for the static dust distribution, and as well general solutions for fluids obeying linear and polytropic state equations. For a constant density distribution, we show that the corresponding solution exhibits the same physical properties as its $3+1$ counterpart with the cosmological constant, details are given Sec. V. One can guess that solutions governed by linear or polytropic state equations should behave like their $3+1$ relatives, of course, having in mind physical content modifications due to the change of dimensions in a similar fashion, as this takes place in the relationship between $2+1$ with $3+1$ isotropic cosmologies. Nevertheless, since most of the perfect fluid solutions of the kind we are interested in are numerically defined, the establishing of correspondences becomes a difficult task; considerations in this respect are left for a future work.

For vanishing cosmological constant, the expression of the pressure (8) is

$$\kappa p(r)=n_1\frac{G(r)}{N(r)}, \tag{10}$$

from which it becomes apparent that the corresponding solution represents a cosmological spacetime; there is no surface of vanishing pressure.

For vanishing λ and zero pressure, the situation slightly changes: the function N becomes a constant, and the corresponding metric can be written as

$$ds^2=-dt^2+\frac{dr^2}{C-2\kappa\int^r r\rho(r)dr}+r^2d\theta^2 \tag{11}$$

for any density function ρ. Of course, the choice of ρ is restricted by physically reasonable matter distributions.

III. CANONICAL COORDINATE SYSTEM $\{t,N,\theta\}$

In this section we show that an alternative formulation of our general solution can be achieved in coordinates $\{t,N,\theta\}$. Indeed, from Eq. (6) for the derivative of the function N, in which we are including—without any loss of generality—the constant n_1 $(N/n_1\rightarrow N, n_1 t\rightarrow t)$, one obtains

$$\frac{dN}{r}=\frac{dr}{G(r)}, \tag{12}$$

hence

$$r^2=C_0+2\int^N G dN. \tag{13}$$

To derive G as a function of the new variable N, one uses G_{tt}, Eq. (3), in the form

$$G\,dG=-(\kappa\rho+\lambda)r\,dr=-(\kappa\rho+\lambda)G dN; \tag{14}$$

therefore, one gets

$$G(N)=C_1-\lambda N-\kappa\int^N\rho(N)dN. \tag{15}$$

Substituting this function G into the expression (13) for r, one has

$$H(N):=r^2=C_0+2C_1N-\lambda N^2-2\kappa\int^N\int^N\rho(N)dNdN. \tag{16}$$

Finally, our metric in the new coordinates $\{t,N,\theta\}$ amounts to

$$ds^2=-N^2dt^2+\frac{dN^2}{H(N)}+H(N)d\theta^2, \tag{17}$$

which is characterized by pressure

$$p(N)=\frac{C_1}{\kappa}\frac{1}{N}-\frac{1}{N}\int^N\rho(N)dN, \tag{18}$$

and an arbitrary energy density $\rho(N)$ depending on the variable N; for physically conceivable solutions, both functions p and ρ have to be positive.

The metric (17) together with the function H from Eq. (16) give an alternative representation of our general solution. This representation will be used to derive particular solutions for a given state equation of the form $p=p(\rho)$. In this approach the expression of the pressure (18) plays a central role.

For a completeness and checking purposes we include the Einstein equations for perfect fluid and cosmological constant for metric (17), considering the function H as an arbitrary one. There are only two equations governing the problem, namely,

$$G_{tt}:\frac{d^2H}{dN^2}=-2\lambda-2\kappa\rho(N), \tag{19}$$

$$G_{NN}:\frac{dH}{dN}=-2\lambda N+2\kappa N p(N). \tag{20}$$

It is well known that the vanishing of the Cotton tensor in three dimensions determines locally the class of conformally flat spaces, in the same fashion as the vanishing of the Weyl tensor singles out conformally flat spaces in higher dimensions. The evaluation of the Cotton tensor

$$C_{\mu\nu\sigma}:=-L_{\mu[\nu;\sigma]}, \quad L_{\mu\nu;\sigma}:=R_{\mu\nu;\sigma}-\tfrac{1}{4}g_{\mu\nu}R_{;\sigma} \tag{21}$$

yields only two independent nonvanishing components

$$C_{ttN}=-\frac{1}{4}\kappa N^2\frac{d\rho}{dN},$$

$$C_{\phi\phi N}=-\frac{1}{4}\kappa g_{\phi\phi}\frac{d\rho}{dN}. \tag{22}$$

Therefore, from the above relations, we conclude that the perfect fluid solution with $\rho=\text{const}$ for static circularly symmetric spacetimes is unique. This result can be stated as a theorem: The perfect fluid solution with constant ρ is the only conformally flat static circularly symmetric spacetime for a perfect fluid source with or without the cosmological constant. It is worth noting the similarity of this theorem with the corresponding one formulated for the interior Schwarzschild metric, see [25,26]. We shall return to this spacetime in Sec. V.

Moreover, returning to the original metric (1), one sees that the coordinate transformation to the metric (17) can be accomplished if the Jacobian of the transformation $J=\det(\partial x/\partial x')=G(r)/r\neq 0$. On the other hand, for a given metric the existence of black holes is associated with the possibility of determining their event horizons; in the present context, a surface $r=\text{const}$ determines an event horizon if its normal $\nabla_\mu r$ becomes a null vector, i.e., $\nabla_\mu r\nabla^\mu r=g^{rr}=0$, which in its turn implies $G(r=\text{const})=0$. Thus, the considered coordinate transformation can be accomplished at all points of the studied spacetime except for points belonging to the event horizon. It is worthwhile to mention that the roots $r=r_i, i=1,2,\ldots,$ if any, of the equation $G(r)=0$ could define singular spacetime hypersurfaces $r=r_i=\text{const}$—inner horizons—internal with respect to the event horizon—the outer horizon—we are dealing with. As far as the existence of singularities is concerned, one cannot express a general statement in this respect; each family of solutions has to be treated in its own right. For instance, the perfect fluid $2+1$ analog to the interior $3+1$ perfect fluid Schwarzschild solution is regular everywhere in the domain of definition of its coordinates, and hence it does not exhibit a singularity at the origin $r=0$ of the coordinate system. To avoid any confusion in determining the existence of singular sets of points, one commonly uses the general procedure of studying the behavior of the geodesics of test particles and light rays to establish their incompleteness.

A. 2+1 perfect fluid solutions for a linear law $p=\gamma\rho$

Although in the previous section we provided the general solution to the posed question of finding all solutions for the circularly symmetric static metric in $2+1$ gravity coupled to a perfect fluid in the presence of the cosmological constant, from the physical point of view, even in this lower dimensional spacetime, it is of interest to analyze certain specific cases, for instance, the solution corresponding to a fluid obeying the law $p=\gamma\rho$, or the more complicated case of a polytropic law $p=C\rho^\gamma$.

The starting point in the present study is the linear relation between pressure and energy density,

$$p(N)=\gamma\rho(N). \tag{23}$$

Substituting $p(N)$ from Eq. (18) into this relation, one gets

$$\frac{C_1}{\kappa}-\int^N\rho(N)dN=\gamma N\rho(N). \tag{24}$$

Differentiating this equation with respect to the variable N, one obtains

$$\frac{d}{dN}(N\rho)+\frac{1}{\gamma N}(N\rho)=0, \tag{25}$$

which has as the general integral

$$\rho(N)=C_2\frac{\gamma-1}{\gamma^2}N^{-(\gamma+1)/\gamma}, \tag{26}$$

where C_2 is an integration constant. Since we arrived at $\rho(N)$, Eq. (26), through differentiation, one has to replace the obtained result into relation (24), or equivalently into Eq. (23), to see if there arises any constraint from it:

$$p(N)=\frac{C_1}{\kappa}\frac{1}{N}+\gamma\rho(N)=\gamma\rho(N)\rightarrow C_1=0. \tag{27}$$

In such a manner, we have established that the constant C_1 vanishes. Replacing the function $\rho(N)$ from Eq. (26) into the expression for $H(N)$, Eq. (16), and accomplishing the integration, one arrives at

$$H(N)=C_0-\lambda N^2+2\kappa C_2N^{(\gamma-1)/\gamma}=r^2. \tag{28}$$

Thus, the metric for a perfect fluid fulfilling a linear state equation in coordinates $\{t,N,\theta\}$ is given by

$$ds^2=-N^2dt^2+\frac{dN^2}{C_0-\lambda N^2+2\kappa C_2N^{(\gamma-1)/\gamma}}+(C_0-\lambda N^2+2\kappa C_2N^{(\gamma-1)/\gamma})d\theta^2. \qquad (29)$$

To express this solution in terms of the radial variable r, one has to be able to solve the algebraic equation (28), in general a transcendent one, for $N=N(r)$.

The evaluation of the Cotton tensor leads to two independent nonvanishing components:

$$C_{ttN}=C_2\frac{\kappa}{4}\frac{(\gamma^2-1)}{\gamma^3}N^{-1/\gamma},$$

$$C_{\phi\phi N}=C_2\frac{\kappa}{4}\frac{(\gamma^2-1)}{\gamma^3}N^{-(2\gamma+1)/\gamma}g_{\phi\phi}. \qquad (30)$$

Some examples of physical interest are described by the treated state equation, for instance: dust, $\gamma=0$; stiff matter, $\gamma=1$; pure radiation, $\gamma=1/2$; and incoherent radiation, $\gamma=1/3$, according to the terminology in use. The reader may encounter details in [27].

B. 2+1 perfect fluid solutions for a polytropic law $p=C\rho^\gamma$

This section is devoted to the derivation of all solutions obeying the polytropic law

$$p=C\rho^\gamma. \qquad (31)$$

Using again the expression of $p(N)$ from Eq. (18), the above polytropic relation can be written as

$$\frac{C_1}{\kappa}-\int^N\rho(N)dN=C\,N\rho^\gamma(N). \qquad (32)$$

Differentiating with respect to N, one obtains

$$-\rho=C\frac{d}{dN}(N\rho^\gamma), \qquad (33)$$

which, by introducing the auxiliary function $Z=N^{1/\gamma}\rho$, can be written as

$$d(Z^{\gamma-1})+\frac{1}{C}d(N^{(\gamma-1)/\gamma})=0, \qquad (34)$$

therefore

$$d\left[\left(\rho^{\gamma-1}+\frac{1}{C}\right)N^{(\gamma-1)/\gamma}\right]=0. \qquad (35)$$

Consequently, the general integral of the studied equation becomes

$$\rho=C^{-1/\gamma}N^{-1/\gamma}[B-C^{-1/\gamma}N^{(\gamma-1)/\gamma}]^{1/(\gamma-1)}, \qquad (36)$$

where B is an integration constant. Entering this ρ into Eq. (31), taking into account that the integral of the density ρ amounts to

$$\int^N\rho(N)dN=-\int^N d[B-C^{-1/\gamma}N^{(\gamma-1)/\gamma}]^{\gamma/(\gamma-1)}, \qquad (37)$$

one arrives at

$$p(N)=\frac{n_1}{\kappa}\frac{C_1}{N}+C\rho^\gamma=C\rho^\gamma\rightarrow C_1=0. \qquad (38)$$

Considering that the first integral of ρ is given by Eq. (37), the expression of the structural function $H(N)$ becomes

$$H=C_0-\lambda N^2+2\kappa\int^N[B-C^{-1/\gamma}N^{(\gamma-1)/\gamma}]^{\gamma/(\gamma-1)}dN=r^2. \qquad (39)$$

Notice that the mentioned integral can be expressed in terms of hypergeometric functions; hence,

$$H(N)=C_0-\lambda N^2+2\kappa B^{\gamma/(\gamma-1)}NF\left(\left[\frac{\gamma}{\gamma-1},-\frac{\gamma}{\gamma-1}\right],\left[\frac{\gamma}{\gamma-1}+1\right],N^{(\gamma-1)/\gamma}C^{-1/\gamma}B^{-1}\right). \qquad (40)$$

Summarizing, in the case of a polytropic equation of state, the general solution is given by the metric (17) with $H(N)$ from Eq. (39), and is characterized by energy density and pressure of the form

$$p=\frac{1}{N}[B-C^{-1/\gamma}N^{(\gamma-1)/\gamma}]^{\gamma/(\gamma-1)},$$

$$\rho=C^{-1/\gamma}N^{-1/\gamma}[B-C^{-1/\gamma}N^{(\gamma-1)/\gamma}]^{1/(\gamma-1)}. \qquad (41)$$

Incidentally, for zero λ, the derivation and study of static circularly symmetric cosmological spacetimes, coupled to perfect fluids fulfilling the polytropic law, was accomplished in [15], where Robertson-Walker cosmologies have also been discussed.

The nonvanishing independent components of the Cotton tensor (21) are

$$C_{ttN}=\frac{1}{4}\kappa N^2\frac{d^2}{dN^2}[C_1-C^{-1/\gamma}N^{(\gamma-1)/\gamma}]^{\gamma/(\gamma-1)},$$

$$C_{\phi\phi N}=\frac{1}{4}\kappa g_{\phi\phi}\frac{d^2}{dN^2}[C_1-C^{-1/\gamma}N^{(\gamma-1)/\gamma}]^{\gamma/(\gamma-1)}. \qquad (42)$$

These polytropic fluids contain, amongst others, certain physically relevant samples: nonrelativistic degenerate fermions, $\gamma=2$; nonrelativistic matter, $\gamma=3/2$; and monatomic and diatomic gases.

IV. OPPENHEIMER-VOLKOV EQUATION

Although when Einstein equations have been fulfilled, the energy-momentum conservation law trivially holds, it is of interest to establish certain properties arising from the Oppenheimer-Volkov equation; see, for instance, [16] in 2 +1 gravity. An alternative derivation of this equation consists in differentiating the Einstein G_{rr}, Eq. (3), with respect to r; this yields

$$\kappa \frac{dp}{dr} = \frac{1}{rN}\frac{dN}{dr}\left(\frac{dG^2}{dr} - \frac{G^2}{r}\right) + \frac{G^2}{r\,N}\left[\frac{d^2N}{dr^2} - \frac{1}{N}\left(\frac{dN}{dr}\right)^2\right]. \tag{43}$$

Substituting the second derivative d^2N/dr^2 from Eq. (4), and the first derivative dN/dr from the G_{rr} equation into Eq. (43), one arrives at the Oppenheimer-Volkov equation

$$\frac{dp}{dr} = -\frac{r}{G^2}(\kappa p - \lambda)(\rho + p). \tag{44}$$

At the circle $r=a$ of vanishing pressure $p(a)=0$, the pressure gradient amounts to

$$\left.\frac{dp}{dr}\right|_{r=a} = \frac{\lambda a}{G(a)^2}\rho(a). \tag{45}$$

Since inside the circle the pressure is positive, $p(r<a)>0$, hence at the circle $r=a$ the pressure gradient has to be nonpositive; consequently the cosmological constant ought to be negative, $\lambda = -1/l^2 < 0$. We shall continue to use λ instead of $-1/l^2$, keeping in mind that λ is a negative constant.

The definition of the mass contained in the circle of radius a is given by

$$M := 2\pi \int_0^a \rho(r) r\, dr, \tag{46}$$

and since the metric components $g_{rr} = 1/G(r)^2$ have to be positive in the domain of definition of the solution, there exists an upper limit for the mass, namely,

$$M \leqslant \frac{\pi}{\kappa}(C - \lambda a^2). \tag{47}$$

Assuming that a state equation $p = p(\rho)$ holds, the matter content is said to be microscopically stable if $dp/d\rho \geqslant 0$. Since Eq. (44) can be written as

$$\frac{dp}{d\rho} = -\frac{r}{G^2}(\kappa p - \lambda)(\rho + p)/\frac{d\rho}{dr}, \tag{48}$$

one concludes that for a microscopically stable fluid with positive pressure p and positive density ρ, this density occurs to be monotonically decreasing, $d\rho/dr < 0$. Moreover, the physical requirement that the speed of sound is less than the velocity of light imposes an upper limit on $dp/d\rho \leqslant 1$.

For our general solution in coordinates $\{t, N, \theta\}$, metric (17), from the expression (18) for the pressure, one establishes

$$\frac{dp}{d\rho} = -\frac{1}{N}(\rho + p) \Big/ \frac{d\rho}{dN}; \tag{49}$$

therefore the density is monotonically decreasing ($d\rho/dN < 0$) if the matter is microscopically stable ($dp/d\rho \geqslant 0$), and vice versa.

Moreover, our fluids, fulfilling the state equation $p = \gamma\rho$, $\gamma > 0$, as well as those obeying the polytropic law $p = C\rho^\gamma$, $C > 0, \gamma > 0$, are microscopically stable fluids.

V. 2+1 PERFECT FLUID SOLUTION WITH CONSTANT ρ

As we demonstrated in Sec. II, for constant ρ the Cotton tensor vanishes, and consequently the corresponding conformally flat space occurs to be unique.

In this section it is shown that one can achieve a full correspondence of the metrics and structural functions for constant energy density perfect fluids in 2+1 and 3+1 gravities. By an appropriate choice of the constant densities and cosmological constants, via a dimensional reduction (freezing of one of the spatial coordinates of the 3+1 spacetime), one obtains the 2+1 metric structure from the 3+1 solution. To achieve the mentioned purpose, the conformally flat static spherically symmetric perfect fluid 3+1 solution with the cosmological constant is presented in a form which allows a comparison with the static circularly symmetric perfect fluid with the λ term and constant ρ of the 2+1 gravity.

In the canonical coordinate system $\{t, N, \theta\}$, for $\rho = \text{const}$, the metric, the expression of the function H, which in its turn establishes the relation to the radial coordinate r, and the pressure are given by

$$ds^2 = -N^2 dt^2 + \frac{dN^2}{H} + H d\theta^2, \tag{50}$$

$$H = C_0 + 2C_1 N - (\lambda + \kappa\rho_0)N^2 = r^2, \tag{51}$$

$$p = -\rho_0 + \frac{C_1}{\kappa}\frac{1}{N}. \tag{52}$$

This unfamiliar looking solution can be given in terms of the radial variable r by expressing N as function of r, $N = N(r)$.

Having in mind the comparison of the 2+1 constant ρ perfect fluid with its 3+1 relative—the Schwarzschild interior solution—we shall derive it from the very beginning by integrating the Einstein equations (3) in coordinates $\{t, r, \theta\}$.

For $\rho = \text{const}$, the integral of Eq. (5) gives

$$G(r) = \sqrt{C - (\kappa\rho + \lambda)r^2}. \tag{53}$$

Substituting $G(r)$ from Eq. (53) into Eq. (7), one obtains

$$N(r) = n_0 - \frac{n_1}{\lambda + \kappa\rho}G(r), \tag{54}$$

which can be written as $N(r)=C_1+C_2G(r)$.

The evaluation of pressure $p(r)$ from Eq. (3) yields

$$\kappa p(r)=\frac{1}{(\kappa\rho+\lambda)N(r)}[n_1\kappa\rho G(r)+n_0\lambda(\kappa\rho+\lambda)], \tag{55}$$

this pressure has to vanish at the boundary $r=a$, which imposes a relation on the constants: $n_0=-n_1\kappa\rho G(a)/[\lambda(\kappa\rho+\lambda)]$, where $G(a)$ is the value of the function $G(r)$ at the boundary, i.e., $G(a)$ is equal to the external value for the $G(r)$ corresponding to the vacuum solution plus λ. A similar comment applies to $N(a)$. Replacing n_0 in Eq. (54), the function $N(r)$ becomes

$$N(r)=-\frac{n_1}{\lambda(\kappa\rho+\lambda)}[\kappa\rho G(a)+\lambda G(r)]. \tag{56}$$

Evaluating $N(r)$ at $r=a$, one comes to $n_1=-\lambda N(a)/G(a)$. Consequently, $N(r)$ amounts to

$$N(r)=\frac{N(a)}{G(a)(\kappa\rho+\lambda)}[\kappa\rho G(a)+\lambda G(r)]. \tag{57}$$

Substituting n_0, n_1, and $N(r)$ into Eq. (55), one gets

$$p(r)=\rho\lambda\frac{G(a)-G(r)}{\kappa\rho G(a)+\lambda G(r)}. \tag{58}$$

Summarizing, the studied perfect fluid for the metric (1) is determined by structural functions $G(r)$ from Eq. (53), and $N(r)$ from Eq. (57), and characterized by a density $\rho=\text{const}$, and pressure p given by Eq. (58).

The $g_{tt}=-N^2$ metric component, with N from Eq. (57), corrects the corresponding one, reported in Ref. [16].

A. 3+1 conformally flat static spherically symmetric perfect fluid solution

In this section we review the main structure of the interior perfect fluid solution in the presence of the cosmological constant λ—the interior Schwarzschild metric with λ—for the $3+1$ static spherically symmetric metric of the form

$$ds^2=-N(r)^2dt^2+\frac{dr^2}{G(r)^2}+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{59}$$

The Einstein equations for a perfect fluid energy-momentum tensor in four dimensions have the same form as those in three dimensions, Eq. (2), except for modifications due to the change of dimensionality, namely, the expression of R now amounts to $R=-\kappa T+4\lambda=\kappa\rho-3\kappa p+4\lambda$. Because the corresponding equations can be found in textbooks, we do not exhibit them here explicitly; we include them for reference in the Appendix.

Since we are interested in conformally flat solutions, we require the vanishing of the conformal Weyl tensor, which for static spherically symmetric spacetime leads to the following equation

$$\frac{d}{dr}\left(\frac{G^2-1}{r^2}\right)=0\rightarrow G(r)=\sqrt{1+c_0r^2}. \tag{60}$$

On the other hand, from the G_{tt} equation, one arrives at

$$G(r)=\sqrt{1-\tfrac{1}{3}(\kappa\rho+\lambda)r^2}; \tag{61}$$

therefore, comparing with Eq. (60), one has $c_0=-(\kappa\rho+\lambda)/3\rightarrow\rho=\text{const}$. Hence, the solution constructed under this condition would correspond to a perfect fluid with $\rho=\text{const}$, named incompressible fluid by Adler *et al.* [28].

Moreover, from $r^2G^2G_{rr}-G_{\theta\theta}=0$, taking into account the form of the function G from Eq. (61), the general expression for $N(r)$ is

$$N(r)=C_1+C_2G(r). \tag{62}$$

The evaluation of the pressure p, from the G_{rr} equation, yields

$$\kappa p(r)=\frac{1}{3N(r)}[C_1(2\lambda-\kappa\rho)-3C_2\kappa\rho G(r)], \tag{63}$$

where $G(r)$ and $N(r)$ are determined in Eqs. (61) and (62), respectively.

This result can be stated in the form of a generalization of the Gürses and Gürsey theorem [29] to the case of nonzero λ: the only conformally flat spherically symmetric static solution to the Einstein equations with the cosmological constant for a perfect fluid is given by the metric (59) with structural functions $G(r)$ and $N(r)$ defined, respectively, by Eqs. (61) and (62). Moreover, replacing in the metric (59) $\sin^2\theta$ by $\sinh^2\theta$ and by θ^2, one obtains, respectively, the pseudospherical and flat branches of solutions.

The constants C_1 and C_2 are determined through the values of structural functions at the boundary $r=a$, where the pressure vanishes, $p(r=a)=0$; they are

$$C_1=3C_2\kappa\rho\frac{G(a)}{2\lambda-\kappa\rho},\quad C_2=\frac{N(a)}{2G(a)}\frac{2\lambda-\kappa\rho}{\lambda+\kappa\rho}, \tag{64}$$

where $G(a)$ is the value of the function $G(r)$ at the boundary $r=a$, i.e., $G(a)$ is equal to the external value of $G(r)$ corresponding to the vacuum plus λ solution. A similar comment applies to $N(a)$. We shall return to this point at the end of this section.

Substituting the expressions of C_1 and C_2 into Eq. (62), one has

TABLE I. Perfect fluid solutions with constant energy density.

3+1 solution	2+1 solution
$ds^2=-N^2dt^2+\frac{dr^2}{G^2}+r^2(d\theta^2+\sin^2\theta d\phi^2)$	$ds^2=-N^2dt^2+\frac{dr^2}{G^2}+r^2d\theta^2$
$G^2=1-\frac{1}{3}(\kappa\rho+\lambda)r^2$	$G^2=C-(\kappa\rho+\lambda)r^2$
$N=\frac{1}{2(\kappa\rho+\lambda)}\frac{N(a)}{G(a)}[3\kappa\rho G(a)+(2\lambda-\kappa\rho)G(r)]$	$N=\frac{1}{(\kappa\rho+\lambda)}\frac{N(a)}{G(a)}[\kappa\rho G(a)+\lambda G(r)]$
$p(r)=\rho(2\lambda-\kappa\rho)\frac{G(a)-G(r)}{3\kappa\rho G(a)+(2\lambda-\kappa\rho)G(r)}$	$p(r)=\rho\lambda\frac{G(a)-G(r)}{\kappa\rho G(a)+\lambda G(r)}$
Kottler	anti−de Sitter $\lambda=-1/l^2$
$G(a)^2=1-\frac{2m}{r}-\frac{1}{3}\lambda r^2$;	$G(a)^2=-M_\infty-\lambda a^2$;
$2m=\kappa\rho a^3/3$	$C=\kappa\rho a^2-M_\infty>0$
$N(a)=G(a)$	$N(a)=G(a)$
$2\lambda_4-\kappa_4\rho_4\to 6\lambda_3,\kappa_4\rho_4\to 2\kappa_3\rho_3$	
$G_4(r)\to G_3(r),N_4(r)\to N_3(r),\kappa_4p_4(r)\to 2\kappa_3p_3(r)$	

$$N(r)=\frac{N(a)}{2G(a)(\kappa\rho+\lambda)}[3\kappa\rho G(a)+(2\lambda-\kappa\rho)G(r)]. \tag{65}$$

Substituting C_1, C_2, and the above expression of $N(r)$ into Eq. (63), one gets

$$p(r)=\rho(2\lambda-\kappa\rho)\frac{G(a)-G(r)}{3\kappa\rho G(a)+(2\lambda-\kappa\rho)G(r)}. \tag{66}$$

For the external Schwarzschild with λ solution, known also the Kottler solution [30], the functions $N(r)$ and $G(r)$ are equal one to another, $N(r)=G(r)$, namely,

$$N(r)=G(r)=\sqrt{1-\frac{2m}{r}-\frac{\lambda}{3}r^2}\quad\text{for } r\geqslant a. \tag{67}$$

Evaluating the mass contained in the sphere of radius a for a constant density ρ, one obtains $2m=\kappa\rho a^3/3$; therefore,

$$N(r)=G(r)=\sqrt{1-\frac{\kappa\rho}{3}\frac{a^3}{r}-\frac{\lambda}{3}r^2}\quad\text{for } r\geqslant a, \tag{68}$$

consequently at $r=a$, one has

$$N(a)=G(a)=\sqrt{1-\frac{\kappa\rho+\lambda}{3}a^2}. \tag{69}$$

In the limit of vanishing cosmological constant, $\lambda=0$, one arrives at the interior Schwarzschild solution.

B. Comparison of perfect fluid solutions

Table I is comparison table of perfect fluid solutions with constant ρ in $2+1$ and $3+1$ gravities. Comparing the structure corresponding to perfect fluid solutions with constant ρ in $3+1$ gravity with the structure of the $2+1$ perfect fluid solution, one arrives at the following correspondence: $2\lambda_4-\kappa_4\rho_4\to 6\lambda_3$, $3\kappa_4\rho_4\to 6\kappa_3\rho_3$, which yields $G_4(r)\to G_3(r),N_4(r)\to N_3(r)$, $\kappa_4p_4(r)\to 2\kappa_3p_3(r)$. Remembering that in $2+1$ gravity there is no Newtonian limit, the choice of κ_3 is free; thus by selecting κ_3 appropriately one can achieve $p_4(r)\to p_3(r)$ and $\rho_4\to\rho_3$.

From this comparison table one can easily conclude that the $2+1$ perfect fluid with constant ρ can be derived from the Schwarzschild interior metric by a simple dimensional reduction: freezing one of the spatial coordinates, say $\phi=\text{const}$, in the $3+1$ solution, one obtains the corresponding $2+1$ spacetime.

Since we accomplished a scaling transformation of the r coordinate, accompanied with the inverse scaling of the angular coordinate θ, one may argue that a conical singularity could arise; one may overcome this trouble by saying that the angular coordinate should be fixed once one brings the $2+1$ metric to the canonical form with $G_3(r)=\sqrt{1-(\kappa\rho+\lambda)r^2}$.

VI. CONCLUDING REMARKS

In this contribution we have derived all perfect fluid solutions for the static circularly symmetric spacetime. The general solution is presented in the standard coordinate system $\{t,r,\theta\}$, and alternatively, in a system—the canonical one—with coordinates $\{t,N,\theta\}$. From the physical point of view, particularly interesting are those fluids fulfilling the linear equation of state, $p=\gamma\rho$, as well as those subjected to the polytropic law $p=C\rho^\gamma$; both families are derived in detail from our general metric referred to as the coordinate system $\{t,N,\theta\}$. Therefore, the derived solutions describe, among others, stiff matter, pure radiation, incoherent radiation, nonrelativistic degenerate fermions, etc. The constant energy density perfect fluid solution with the cosmological constant of the $2+1$ gravity is singled out among all static circular spacetimes as the only conformally flat space—its Cotton tensor vanishes—sharing the conformal flatness property with its $3+1$ counterpart, i.e., the Schwarzschild interior perfect fluid solution with λ; a comparison table for these solutions with constant energy density is included. To our mind, these analytical results can be used in the analysis of physically plausible perfect fluid solutions of the $3+1$ gravity even in the case of numerically derived solutions.

ACKNOWLEDGMENTS

This work was partially supported by the CONACyT, Grant 38495E, and the Sistema Nacional de Investigadores (SNI).

APPENDIX: 3+1 EINSTEIN EQUATIONS WITH λ FOR PERFECT FLUID

The Einstein equations with the cosmological constant for perfect fluids for the metric (59) explicitly amount to

$$G_{tt}=-\frac{N^2}{r^2}\left(r\frac{dG^2}{dr}+G^2-1\right)=N^2(\kappa\rho+\lambda),$$

$$G_{rr}=\frac{1}{G^2Nr^2}\left(2rG^2\frac{dN}{dr}-N+NG^2\right)=\frac{1}{G^2}(\kappa p-\lambda),$$

$$G_{\theta\theta}=\frac{r}{N}\left(G^2\frac{dN}{dr}+\frac{1}{2}N\frac{dG^2}{dr}+rG^2\frac{d^2N}{dr^2}+\frac{r}{2}\frac{dN}{dr}\frac{dG^2}{dr}\right)=r^2(\kappa p-\Lambda),$$

$$G_{\phi\phi}=\sin^2\theta G_{\theta\theta}. \tag{A1}$$

[1] S. Carlip, *Quantum Gravity in 2+1 Dimensions* (Cambridge University Press, Cambridge, England, 1998).
[2] A. Staruszkiewicz, Acta Phys. Pol. **24**, 734 (1963).
[3] S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1984).
[4] G. Clement, Nucl. Phys. **B114**, 437 (1976).
[5] J.R. Gott and M. Alpert, Gen. Relativ. Gravit. **16**, 243 (1984).
[6] S. Deser and P.O. Mazur, Class. Quantum Grav. **2**, L51 (1985).
[7] M. Kamata and T. Koikawa, Phys. Lett. B **353**, 196 (1995).
[8] M. Cataldo and P. Salgado, Phys. Lett. B **448**, 20 (1999).
[9] M. Cataldo and A. García, Phys. Rev. D **61**, 084003 (2000).
[10] M. Cataldo, Phys. Lett. B **529**, 143 (2002).
[11] C. Martínez and J. Zanelli, Phys. Rev. D **54**, 3830 (1996).
[12] J.D. Barrow, A.B. Burd, and D. Lancaster, Class. Quantum Grav. **3**, 551 (1986).
[13] P. Collas, Am. J. Phys. **45**, 833 (1977).
[14] S. Giddings, J. Abbott, and K. Kuchař, Gen. Relativ. Gravit. **16**, 751 (1984).
[15] N.J. Cornish and N.E. Frankel, Phys. Rev. D **43**, 2555 (1991).
[16] N. Cruz and J. Zanelli, Class. Quantum Grav. **12**, 975 (1995).
[17] M. Gürses, Class. Quantum Grav. **11**, 2585 (1994).
[18] M. Cataldo, S. del Campo, and A. García, Gen. Relativ. Gravit. **33**, 1245 (2001).
[19] A.A. García and C. Campuzano, Phys. Rev. D **66**, 124018 (2002).
[20] R. Emparan, G.T. Horowitz, and R.C. Myers, J. High Energy Phys. **01**, 007 (2000).
[21] M. Cataldo, S. del Campo, and A. A. García, "The 2+1 from 3+1 Friedmann-Robertson-Walker Cosmological Models," Report No. cinvestav-gr12Dec02, 2002.
[22] R. C. Tolman, Proc. Natl. Acad. Sci. U.S.A. **20**, 169 (1934); see also H. Stephani, *General Relativity*, 2nd ed. (Cambridge University Press, Cambridge, England, 1990).
[23] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 1981).
[24] E. Cotton, Ann. Fac. Sci. Univ. Toulouse Sci. Math. Sci. Phys. **1**, 385 (1899).
[25] C.D. Collinson, Gen. Relativ. Gravit. **7**, 419 (1976).
[26] A.A. García, Gen. Relativ. Gravit. **20**, 589 (1988).
[27] Ya.B. Zel'dovich and I.D. Novikov, *Stars and Relativity* (Dover, New York, 1996).
[28] R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill, New York, 1965).
[29] M. Gürses and Y. Gürsey, Nuovo Cimento Soc. Ital. Fis., B **25B**, 786 (1975).
[30] F. Kottler, Ann. Phys. (Leipzig) **56**, 410 (1918).